Causal Inference
Theory and Applications in Enterprise Computing

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Agenda

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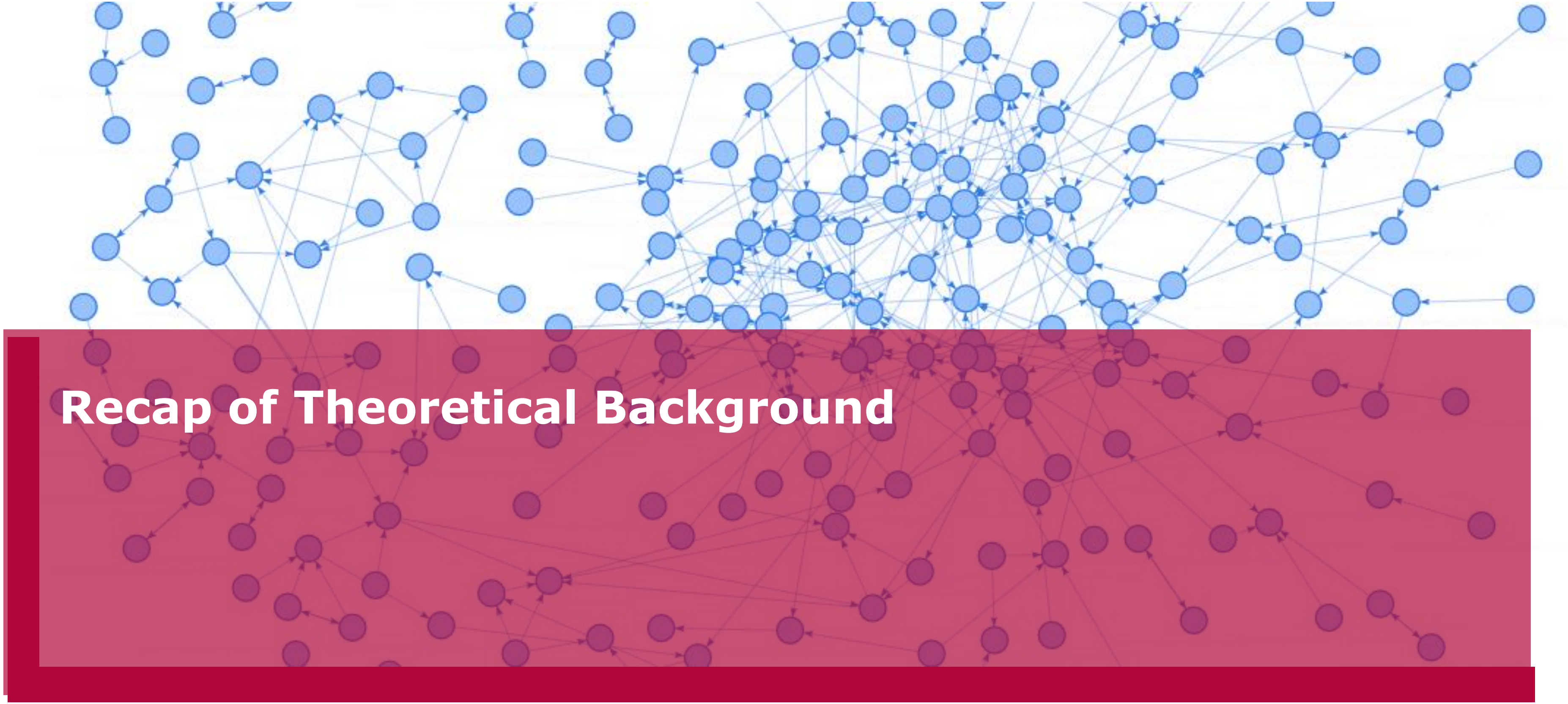
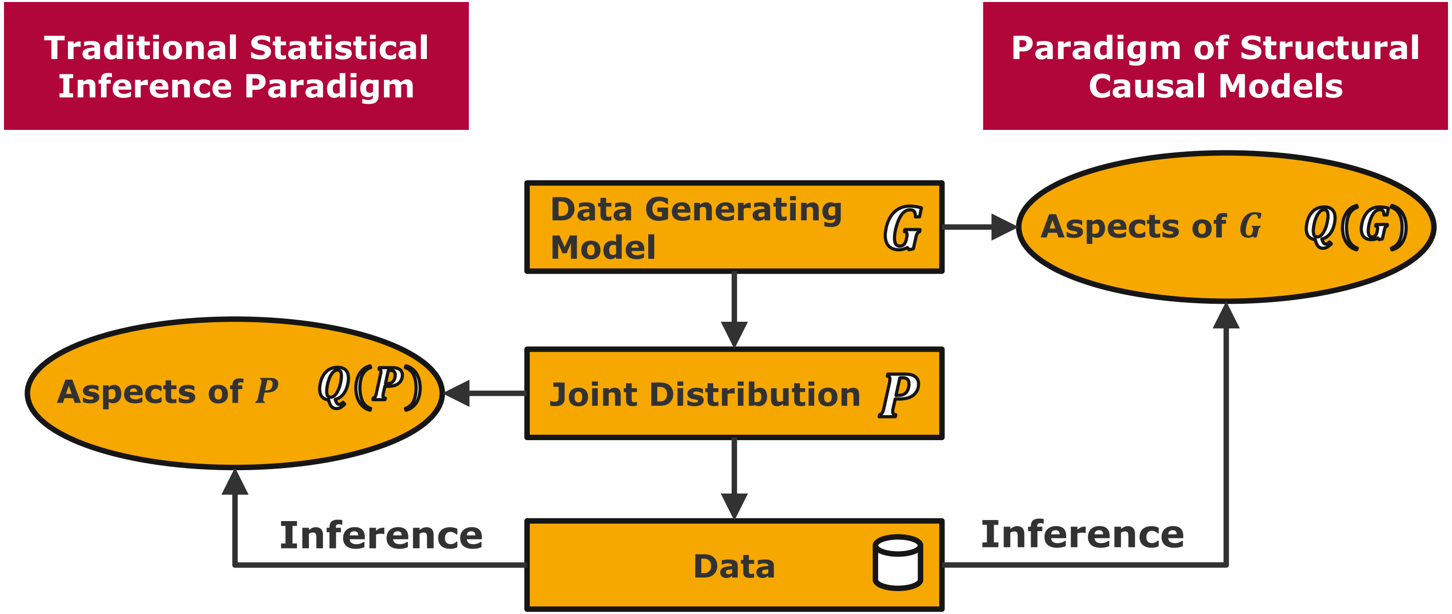
- **Recap of Theoretical Background**
- **Constraint-Based Causal Structure Learning**
    1. Introduction
    2. Constraint-Based Causal Structure Learning
    3. PC Algorithm
    4. PC Algorithm in the Cooling House Example
    5. Extensions of the PC Algorithm
    6. Excursion: Other Causal Structure Learning Concepts
- **Causal Inference in Application**
- **Outlook: Group Work on Research Topics**



Recap of Theoretical Background

Recap of Theoretical Background

Causal Inference in a Nutshell



E.g., what is the sailors' probability of recovery when **we see** a treatment with lemons?

$$Q(P) = P(recovery|lemons)$$

E.g., what is the sailors' probability of recovery if **we do** treat them with lemons?

$$Q(G) = P(recovery|do(lemons))$$

Recap of Theoretical Background

Causal Graphical Models

- Causal Structures formalized by *DAG (directed acyclic graph)* G with random variables $V_1, \dots, V_n$ as vertices.

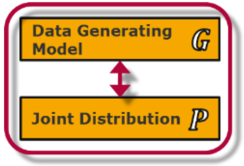
- *Causal Sufficiency*, *Causal Faithfulness* and *Global Markov Condition* imply

$$(X \perp Y \mid Z)_G \Leftrightarrow (X \perp Y \mid Z)_P.$$

- *Local Markov Condition* states that the density $p(\mathrm{v}_1, \dots, v_n)$ then factorizes into

$$p(v_1, \dots, v_n) = \prod_{i=1}^{n} p(v_i | Pa(v_i)).$$

- Causal conditional $p\big(v_j \big| Pa\big(v_j\big)\big)$ represent *causal mechanisms*.



Data Generating Model $\mathbb{G}$

Joint Distribution $\mathbb{P}$

Recap of Theoretical Background

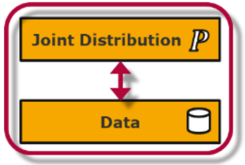
Statistical Inference

- *Null Hypothesis* H_0 is the claim that is initially assumed to be true
- *Alternative Hypothesis* H_1 is a claim that contradicts the H_0
- How to test a hypothesis?
  - Approximate T under H_0 by a known distribution
  - Different distributions yield to different tests, e.g., T-test, χ^2-test, etc.
  - Derive rejection criteria for H_0

$$\left.\begin{array}{l} - \quad c\text{-value: reject } H_0 \text{ if } T(x_n) > \mathrm{c} \text{ for a } c \in \mathbb{R} \\ - \quad p\text{-value: reject } H_0 \text{ if } P_{H_0}\big(T(X) > T(x)\big) < \alpha \end{array}\right\} \text{ are equivalent}$$

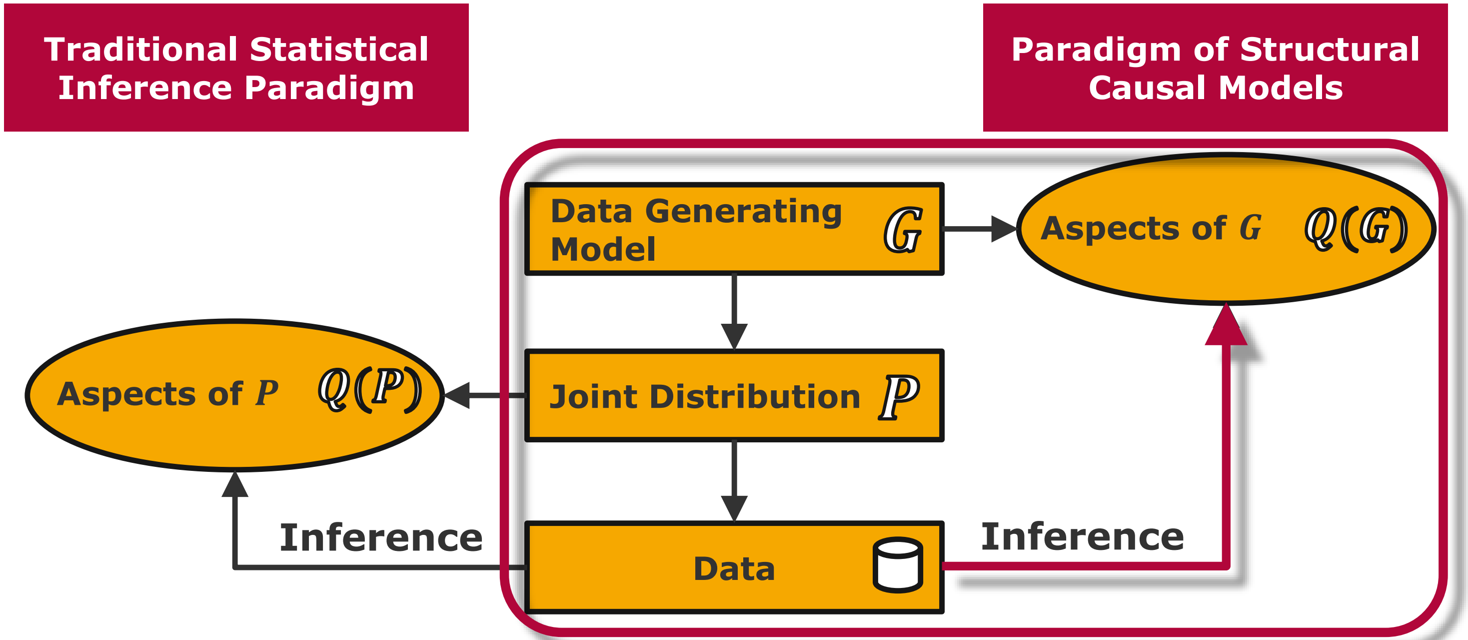
- *(Conditional) Independence Test*

  Distribution of $V_1, \dots, V_N \Rightarrow$ dependence measures $T(V_i, V_j, \boldsymbol{S}) \Rightarrow$ test $H_0\colon t = 0$
- Allows for *constraint-based causal structure learning*



Constraint-Based Causal Structure Learning

1. Introduction
The Concept



Traditional Statistical Inference Paradigm

Paradigm of Structural Causal Models

Data Generating Model G

Aspects of G $Q(G)$

Joint Distribution P

Aspects of P $Q(P)$

Data

Inference

Inference

E.g., what is the sailors’ probability of recovery when **we see** a treatment with lemons?

$$Q(P) = P(recovery|lemons)$$

E.g., what is the sailors’ probability of recovery if **we do** treat them with lemons?

$$Q(G) = P(recovery|do(lemons))$$

1. Introduction

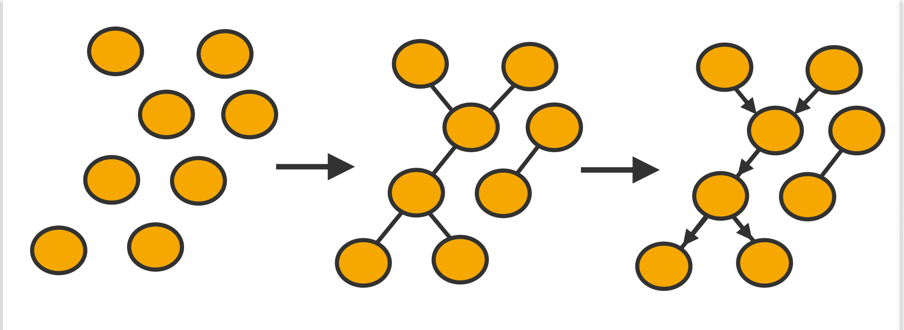
Recap: Basis of Causal Structure Learning (Pearl et al.)

- **Assumptions:**
  - Causal Sufficiency
  - Global Markov Condition
  - Causal Faithfulness

- **Causal Structure Learning:**
  - Accept only those DAG’s G as causal hypothesis for which
    $$(X \perp Y | Z)_G \Leftrightarrow (X \perp Y | Z)_P.$$
  - Identifies causal DAG up to *Markov equivalence class* (DAGs that imply the same conditional independencies)
  - The Markov equivalence class of a DAG G includes all DAGs G' that have the same *skeleton* C and the same v*-structures*
  - Markov equivalence class of the true DAG G that can be uniquely described by a *completed partially directed acyclic graph (CPDAG)*

2. Constraint-Based Causal Structure Learning

Algorithmic Construction (I/II)



Idea:

1. Construct skeleton C
2. Find v-structures
3. Direct further edges that follow from
   - ☐ Graph is acyclic
   - ☐ All v-structures have been found in 2

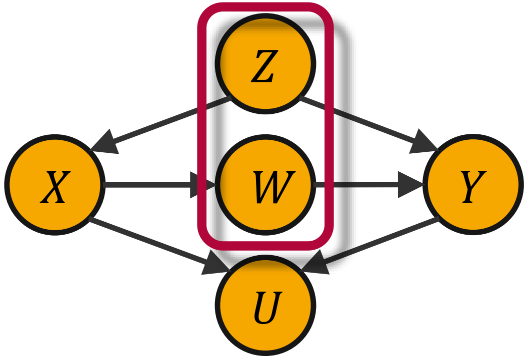
➡ IC algorithm by Verma and Pearl (1990) to reconstruct CPDAG G from P

2. Constraint-Based Causal Structure Learning
Algorithmic Construction (II/II)

Theorem
Assume Markov condition and faithfulness holds. Then X and Y are linked by an edge if and only if there is no set $S(X,Y)$ such that

$$(X \perp Y | \, S(X,Y))_{P.}$$

- I.e., dependence mediated by other variables can be screened off by conditioning on an *appropriate* set



- $X \perp Y \,|\{Z,W\}$
- But not:
  - $X \perp Y \,|\, U$
  - $X \perp Y \,|\, \{Z,W,U\}$

…but not by conditioning on all other variables!

- $S(X,Y)$ is called *separation set of X and Y*

3. PC Algorithm
The Idea

Question:
How to find the appropriate separation sets $S(V_i, V_j)$ for all variables V_i and V_j?

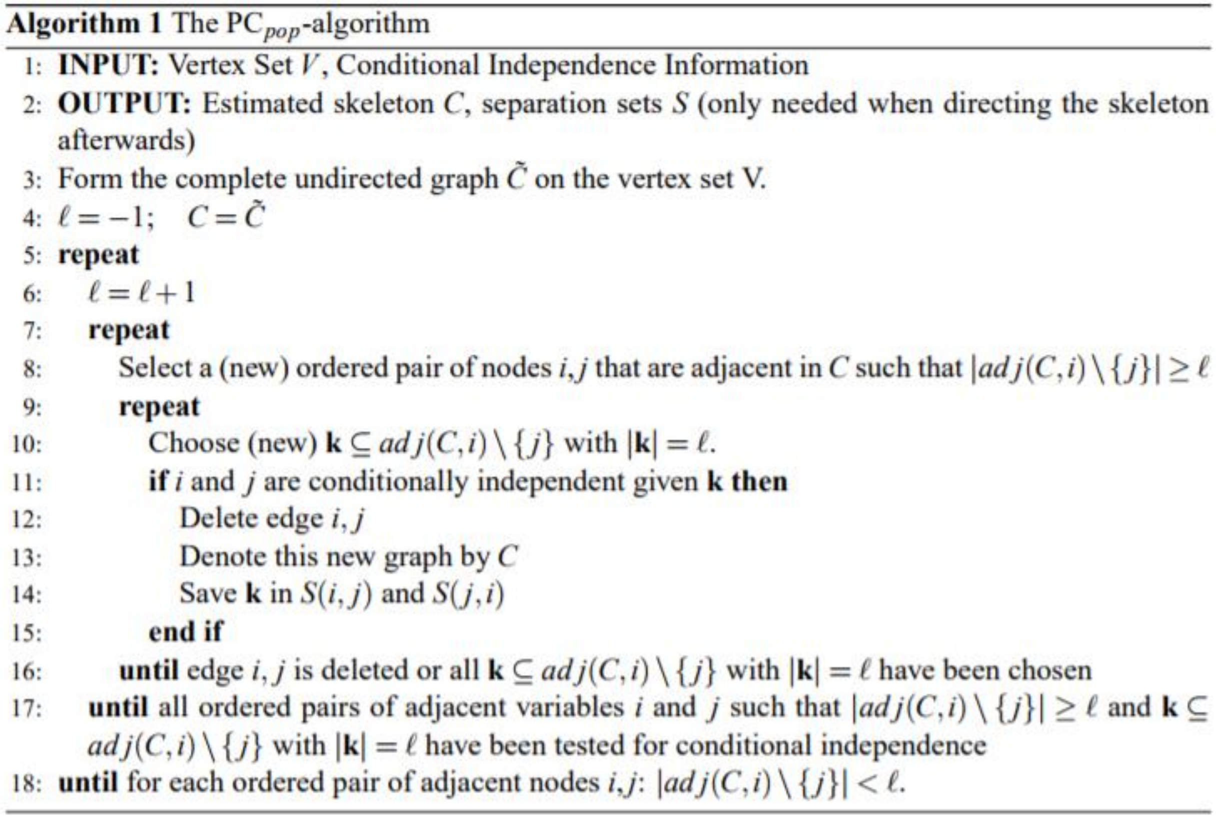
- Check $\mathrm{V_i} \perp V_j \mid S(V_i, V_j)$ for all possible separation sets $S(V_i, V_j) \subseteq V \setminus \{V_i, V_j\}$
  - Computationally infeasible for large V

- Efficient construction of the skeleton C

  Iteration over size of the separation sets S:
  1. Remove all edges $X - Y$ with $X \perp Y$
  2. Remove all edges $X - Y$
     for which there is an adjacent $Z \neq Y$ of X with $X \perp Y \mid Z$
  3. Remove all edges $X - Y$
     for which there are two adjacent $\mathrm{Z_1}, \mathrm{Z_2} \neq Y$ of X with $X \perp Y \mid \{Z_1, Z_2\}$
  4. …

➡ PC algorithm by Spirtes et al. (1993) to reconstruct CPDAG G from P

3. PC Algorithm
Skeleton Discovery: Pseudocode



Algorithm 1 The PC$_{pop}$-algorithm

```
INPUT: Vertex Set $V$, Conditional Independence Information
OUTPUT: Estimated skeleton $C$, separation sets $S$ (only needed when directing the skeleton afterwards)
Form the complete undirected graph $\tilde{C}$ on the vertex set V.
$\ell = -1; \quad C = \tilde{C}$
repeat
    $\ell = \ell + 1$
    repeat
        Select a (new) ordered pair of nodes $i, j$ that are adjacent in $C$ such that $|adj(C,i) \setminus \{j\}| \geq \ell$
        repeat
            Choose (new) $\mathbf{k} \subseteq adj(C,i) \setminus \{j\}$ with $|\mathbf{k}| = \ell$.
            if $i$ and $j$ are conditionally independent given $\mathbf{k}$ then
                Delete edge $i, j$
                Denote this new graph by $C$
                Save $\mathbf{k}$ in $S(i,j)$ and $S(j,i)$
            end if
        until edge $i, j$ is deleted or all $\mathbf{k} \subseteq adj(C,i) \setminus \{j\}$ with $|\mathbf{k}| = \ell$ have been chosen
    until all ordered pairs of adjacent variables $i$ and $j$ such that $|adj(C,i) \setminus \{j\}| \geq \ell$ and $\mathbf{k} \subseteq adj(C,i) \setminus \{j\}$ with $|\mathbf{k}| = \ell$ have been tested for conditional independence
until for each ordered pair of adjacent nodes $i, j$: $|adj(C,i) \setminus \{j\}| < \ell$.
```

3. PC Algorithm

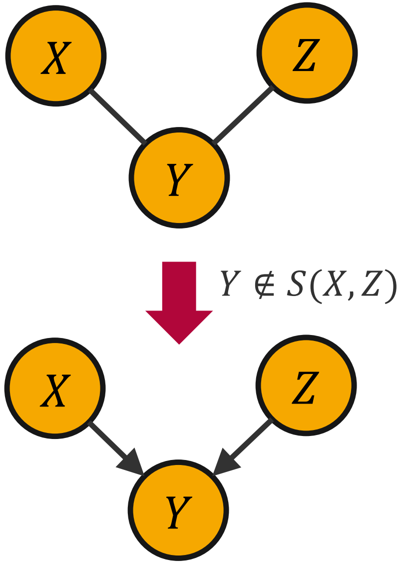
Edge Orientation: v-Structures

- Assume the skeleton is given by:
  - Given $X - Y - Z$ with X and Z nonadjacent
  - Given $S(X,Z)$ with $X \perp Z \mid S(X,Z)$

- A priori, there are 4 possible orientations

$$\left.\begin{array}{l} \square \quad X \rightarrow Y \rightarrow Z \\ \square \quad X \leftarrow Y \rightarrow Z \\ \square \quad X \leftarrow Y \leftarrow Z \end{array}\right\} Y \in S(X,Z)$$

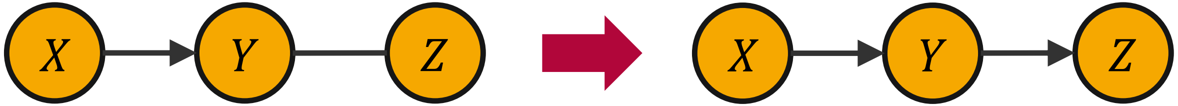
$$\left.\begin{array}{l} \square \quad X \rightarrow Y \leftarrow Z \end{array}\right\} Y \notin S(X,Z)$$



v-Structures:
If $Y \notin S(X,Z)$ then replace $X - Y - Z$ by $X \rightarrow Y \leftarrow Z$.

3. PC Algorithm

Edge Orientation: Rule 1

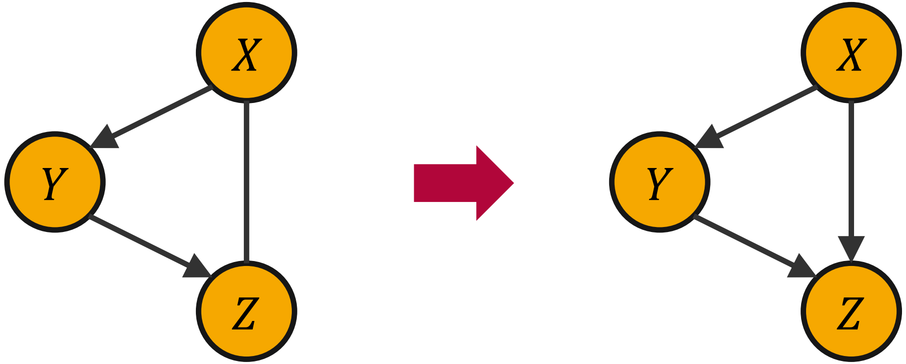


(Otherwise we get a new v-structure)

Rule 1:
Orient $Y - Z$ to $Y \rightarrow Z$ whenever
there is an arrow $X \rightarrow Y$ s.t. X and Z are nonadjacent

3. PC Algorithm

Edge Orientation: Rule 2

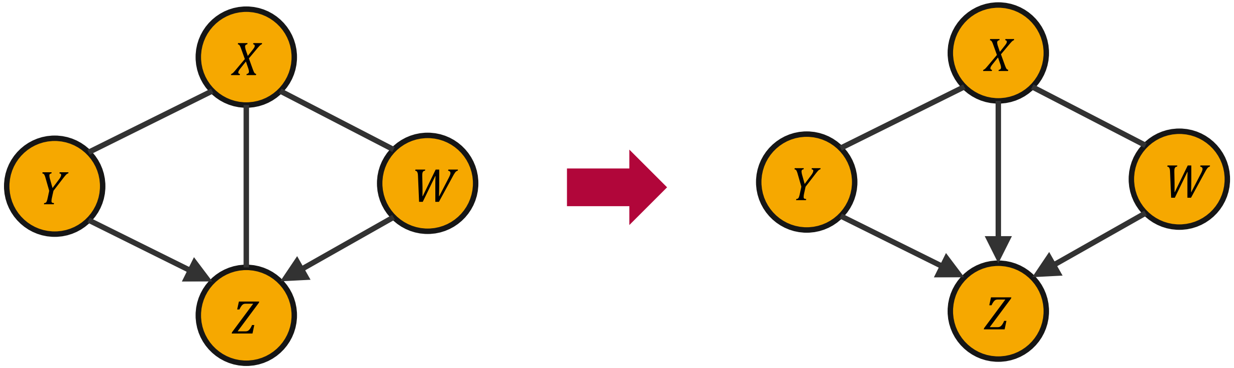


(Otherwise we get a cycle)

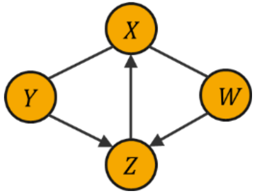
Rule 2:
Orient $X - Z$ to $X \to Z$ whenever
there is a chain $X \to Y \to Z$

3. PC Algorithm

Edge Orientation: Rule 3



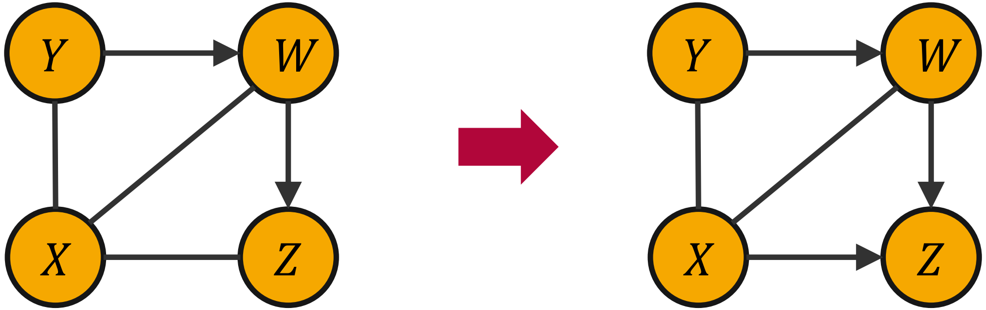
(Could not be completed without creating a cycle or a new v-structure)



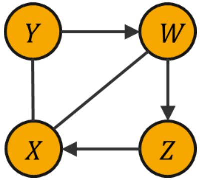
Rule 3:
Orient $X - Z$ to $X \rightarrow Z$ whenever
there are two chains $X - Y \rightarrow Z$ and $X - W \rightarrow Z$ s.t. Y and W are nonadjacent

3. PC Algorithm

Edge Orientation: Rule 4



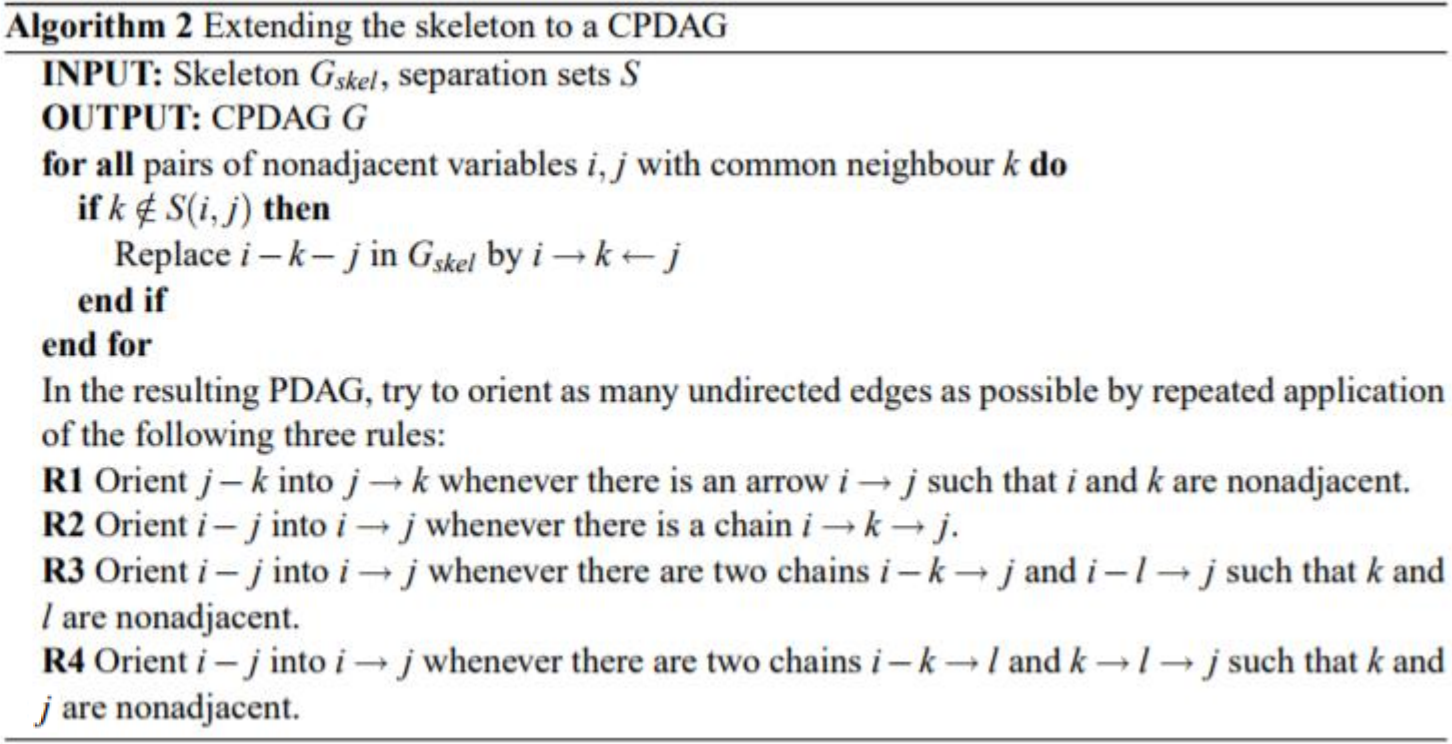
(Could not be completed without creating a cycle or a new v-structure)



Rule 4:
Orient $X - Z$ to $X \to Z$ whenever
there are two chains $X - Y \to W$ and $Y \to W \to Z$ s.t. Y and Z are nonadjacent

3. PC Algorithm

Edge Orientation: Pseudocode



Algorithm 2 Extending the skeleton to a CPDAG

```
INPUT: Skeleton $G_{skel}$, separation sets $S$
OUTPUT: CPDAG $G$
for all pairs of nonadjacent variables $i, j$ with common neighbour $k$ do
    if $k \notin S(i, j)$ then
        Replace $i - k - j$ in $G_{skel}$ by $i \rightarrow k \leftarrow j$
    end if
end for
In the resulting PDAG, try to orient as many undirected edges as possible by repeated application of the following three rules:
R1 Orient $j - k$ into $j \rightarrow k$ whenever there is an arrow $i \rightarrow j$ such that $i$ and $k$ are nonadjacent.
R2 Orient $i - j$ into $i \rightarrow j$ whenever there is a chain $i \rightarrow k \rightarrow j$.
R3 Orient $i - j$ into $i \rightarrow j$ whenever there are two chains $i - k \rightarrow j$ and $i - l \rightarrow j$ such that $k$ and $l$ are nonadjacent.
R4 Orient $i - j$ into $i \rightarrow j$ whenever there are two chains $i - k \rightarrow l$ and $k \rightarrow l \rightarrow j$ such that $k$ and $j$ are nonadjacent.
```

3. PC Algorithm

A Review

Advantages

- Testing all sets $S(X,Y)$ containing the adjacencies of X is sufficient
- Many edges can be removed already for small sets
- Depending on sparseness, the algorithm only requires independence tests with small conditioning sets $S(X,Y)$
- Polynomial complexity for graph of N vertices of bounded degree k, i.e.,

$$\frac{N^2(N-1)^{k-1}}{(k-1)!}$$

- Asymptotic consistency (under technical assumptions), i.e.,

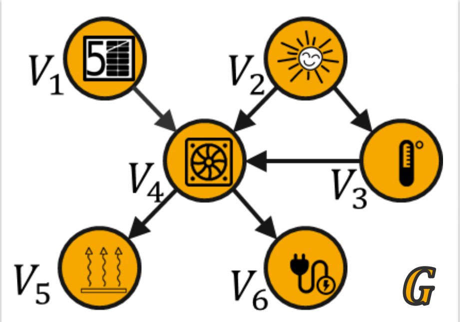
$$\Pr\left(\hat{G}=G\right)\to 1 \quad (n\to\infty)$$

Disadvantages

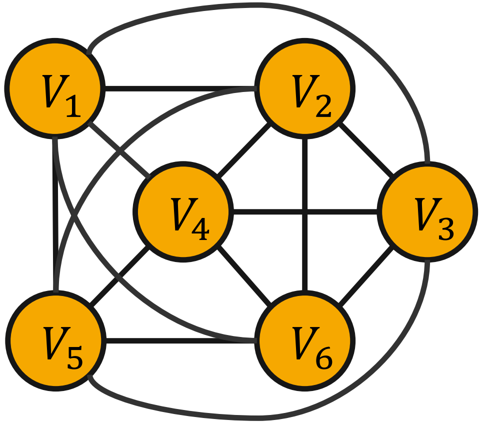
- In the worst case, complexity exponential to number of vertices N
- Assumes causal sufficiency, faithfulness and Markov conditions

4. PC Algorithm in the Cooling House Example
Cooling House Example (I/V)

- Assume the true DAG G is given by:



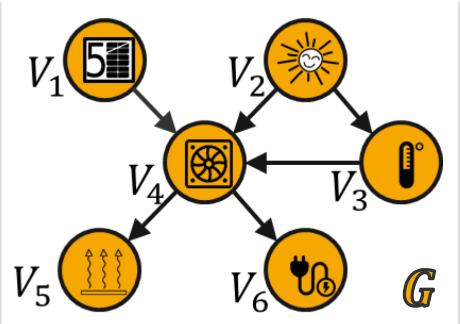
- We start with a fully connected undirected graph:



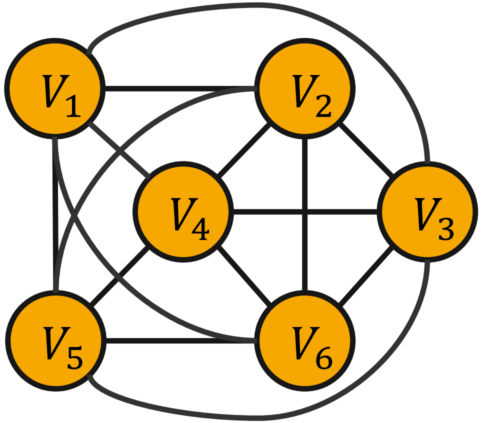
4. PC Algorithm in Application

Cooling House Example (II/V)

- Assume the true DAG G is given by:



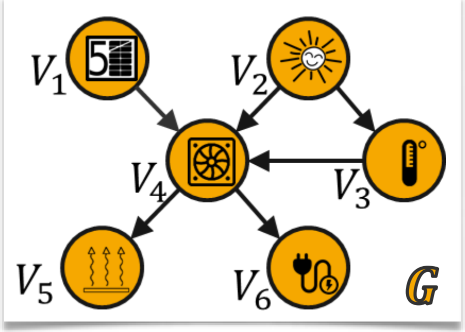
- Remove all edges $X - Y$ that are directly independent, i.e., $X \perp Y \mid \emptyset$
  - $V_1 \perp V_2$
  - $V_1 \perp V_3$



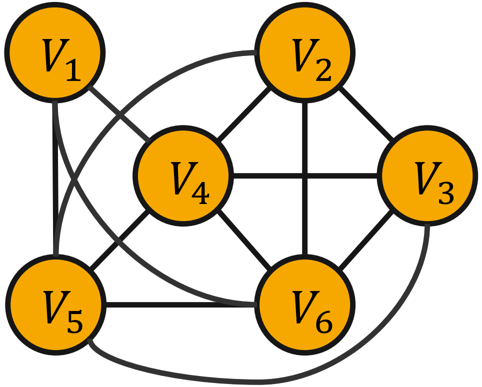
4. PC Algorithm in Application

Cooling House Example (III/V)

- Assume the true DAG G is given by:



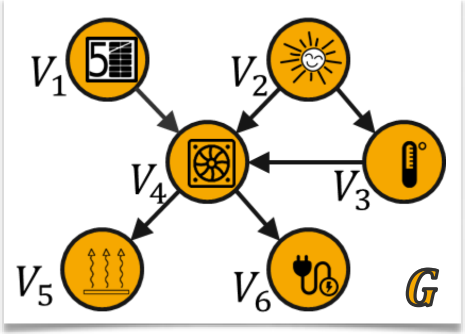
- Remove all edges $X - Y$ having separation sets of size 1, i.e., $X \perp Y \mid Z$
  - $V_1 \perp V_5 \mid V_4$
  - $V_1 \perp V_6 \mid V_4$
  - $V_2 \perp V_5 \mid V_4$
  - $V_2 \perp V_6 \mid V_4$
  - $V_3 \perp V_5 \mid V_4$
  - $V_3 \perp V_6 \mid V_4$
  - $V_5 \perp V_6 \mid V_4$



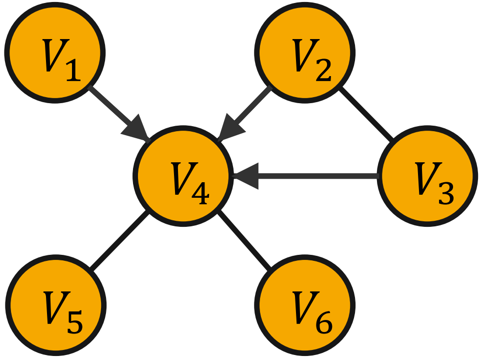
4. PC Algorithm in Application

Cooling House Example (IV/V)

- Assume the true DAG G is given by:



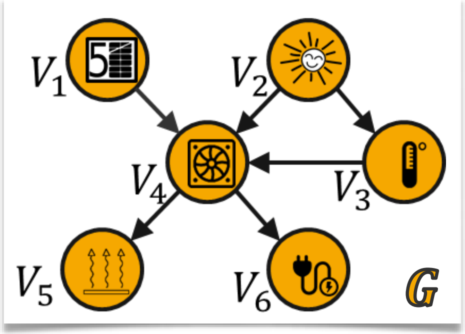
- Find v-structures, i.e., orient $X - Y - Z$ to $X \rightarrow Y \leftarrow Z$ if $Y \notin S(X,Z)$
    - $V_4 \notin S(V_1, V_2)$
    - $V_4 \notin S(V_1, V_3)$



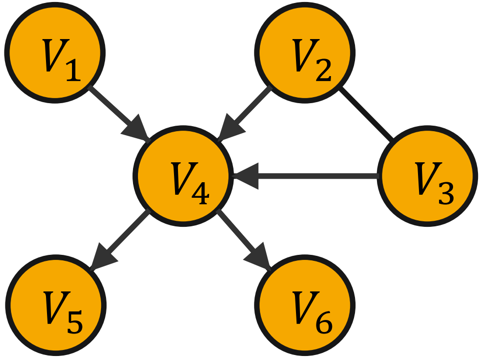
4. PC Algorithm in Application

Cooling House Example (V/V)

- Assume the true DAG G is given by:



- Orient further edges (such that no further v-structures arise)
  - $V_1 \rightarrow V_4 - V_5$ (Rule 1)
  - $V_1 \rightarrow V_4 - V_6$ (Rule 1)



- No further edges can be oriented, i.e., $V_2 - V_3$ remain undirected

5. Extensions of the PC Algorithm
Order Independence (Colombo et al. 2014)

PC algorithm

Order of $V_1, \dots, V_N$ affects estimation of

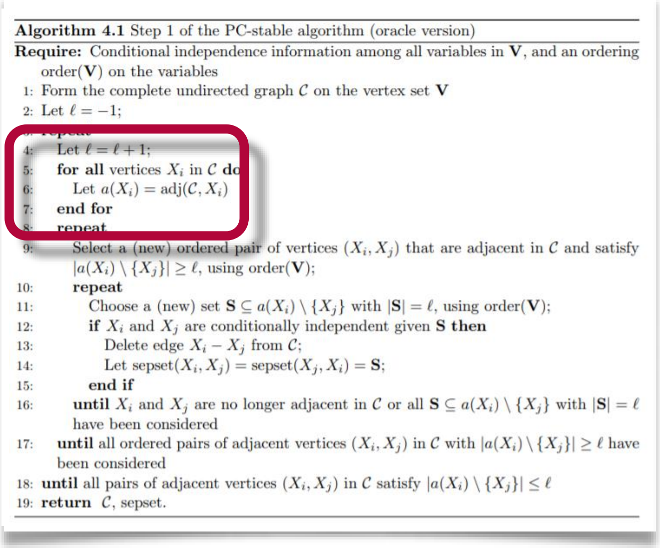
1. Skeleton C
2. Separating sets $S(V_i, V_j)$
3. Edge orientation

PC-stable algorithm

For each level l

- Compute and store the adjacency set $a(V_i)$ of all vertices V_i
- Use $a(V_i)$ for search of separation sets

⇨ Edge deletion longer affects which conditional independencies are checked for other pairs of variables at this level l



Algorithm 4.1 Step 1 of the PC-stable algorithm (oracle version)

```
Require: Conditional independence information among all variables in $\mathbf{V}$, and an ordering
    order($\mathbf{V}$) on the variables
Form the complete undirected graph $\mathcal{C}$ on the vertex set $\mathbf{V}$
Let $\ell = -1$;
repeat
    Let $\ell = \ell + 1$;
    for all vertices $X_i$ in $\mathcal{C}$ do
        Let $a(X_i) = \text{adj}(\mathcal{C}, X_i)$
    end for
    repeat
        Select a (new) ordered pair of vertices $(X_i, X_j)$ that are adjacent in $\mathcal{C}$ and satisfy
        $|a(X_i) \setminus \{X_j\}| \geq \ell$, using order($\mathbf{V}$);
        repeat
            Choose a (new) set $\mathbf{S} \subseteq a(X_i) \setminus \{X_j\}$ with $|\mathbf{S}| = \ell$, using order($\mathbf{V}$);
            if $X_i$ and $X_j$ are conditionally independent given $\mathbf{S}$ then
                Delete edge $X_i - X_j$ from $\mathcal{C}$;
                Let sepset$(X_i, X_j)$ = sepset$(X_j, X_i) = \mathbf{S}$;
            end if
        until $X_i$ and $X_j$ are no longer adjacent in $\mathcal{C}$ or all $\mathbf{S} \subseteq a(X_i) \setminus \{X_j\}$ with $|\mathbf{S}| = \ell$
        have been considered
    until all ordered pairs of adjacent vertices $(X_i, X_j)$ in $\mathcal{C}$ with $|a(X_i) \setminus \{X_j\}| \geq \ell$ have
    been considered
until all pairs of adjacent vertices $(X_i, X_j)$ in $\mathcal{C}$ satisfy $|a(X_i) \setminus \{X_j\}| \leq \ell$
return $\mathcal{C}$, sepset.
```

5. Extensions of the PC Algorithm

Parallelization (Le et al. 2016)

PC algorithm

Limitations:

1. Order-dependent (→*PC-stable*)
2. Sequential execution does not utilize modern hardware

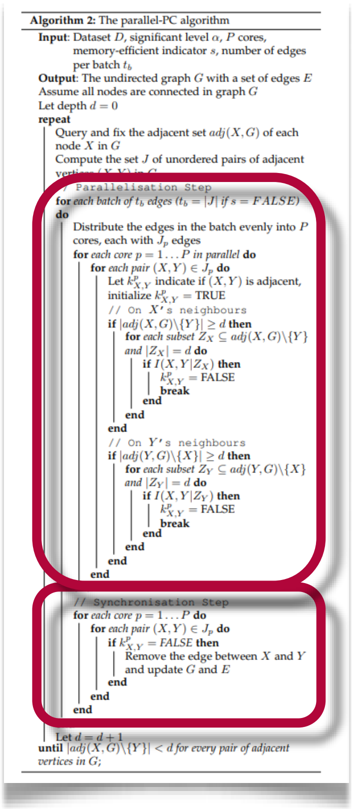
⇨ Long runtime hinders its application on high dimensional datasets

parallelPC algorithm

PC-stable allows for easy parallelization at each level l, i.e.,

1. CI tests are distributed evenly among the cores
2. Each core performs its own sets of CI tests in parallel with the others
3. Synchronize test results into the global skeleton C

⇨ Efficient in high dimensional datasets and consistent with PC-stable algorithm



```
Algorithm 2: The parallel-PC algorithm
  Input: Dataset $D$, significant level $\alpha$, $P$ cores,
         memory-efficient indicator $s$, number of edges
         per batch $t_b$
  Output: The undirected graph $G$ with a set of edges $E$
  Assume all nodes are connected in graph $G$
  Let depth $d = 0$
  repeat
      Query and fix the adjacent set $adj(X, G)$ of each
      node $X$ in $G$
      Compute the set $J$ of unordered pairs of adjacent
      vertices $(X, Y)$ in $G$
      // Parallelisation Step
      for each batch of $t_b$ edges ($t_b = |J|$ if $s = FALSE$)
      do
          Distribute the edges in the batch evenly into $P$
          cores, each with $J_p$ edges
          for each core $p = 1 \ldots P$ in parallel do
              for each pair $(X, Y) \in J_p$ do
                  Let $k^p_{X,Y}$ indicate if $(X, Y)$ is adjacent,
                  initialize $k^p_{X,Y}$ = TRUE
                  // On $X$'s neighbours
                  if $|adj(X, G)\backslash\{Y\}| \geq d$ then
                      for each subset $Z_X \subseteq adj(X, G)\backslash\{Y\}$
                      and $|Z_X| = d$ do
                          if $I(X, Y | Z_X)$ then
                              $k^p_{X,Y}$ = FALSE
                              break
                          end
                      end
                  end
                  // On $Y$'s neighbours
                  if $|adj(Y, G)\backslash\{X\}| \geq d$ then
                      for each subset $Z_Y \subseteq adj(Y, G)\backslash\{X\}$
                      and $|Z_Y| = d$ do
                          if $I(X, Y | Z_Y)$ then
                              $k^p_{X,Y}$ = FALSE
                              break
                          end
                      end
                  end
              end
          end
      // Synchronisation Step
      for each core $p = 1 \ldots P$ do
          for each pair $(X, Y) \in J_p$ do
              if $k^p_{X,Y} = FALSE$ then
                  Remove the edge between $X$ and $Y$
                  and update $G$ and $E$
              end
          end
      end
      Let $d = d + 1$
  until $|adj(X, G)\backslash\{Y\}| < d$ for every pair of adjacent
  vertices in $G$;
```

5. Extensions of the PC Algorithm

Theoretical Extensions (A Selection)

- **Weaker form of faithfulness**
  - ☐ Learn a Markov equivalence class of DAGs under a weaker-than-standard causal faithfulness assumption
  - ☐ Assumes Adjacency-Faithfulness to justify the step of recovering adjacencies in constraint-based algorithms
  - ⇨ Conservative PC (CPC) by Ramsey et al. (1995)
- **Allow for latent and selection variables**
  - ☐ Learn a Markov equivalence class of DAGs with latent and selection variables
  - ☐ Follows maximal ancestral graph (MAG) models
  - ⇨ Fast causal inference (FCI) by Spirtes et al. (1999)
- **Allow for cycles**
  - ☐ Learn Markov equivalence classes of directed (not necessarily acyclic) graphs under the assumption of causal sufficiency.
  - ⇨ Cyclic causal discovery (CCD) by Richardson (1996)

6. Excursion:
Other Causal Structure Learning Concepts

Score-based methods

- *"search-and-score approach"*, i.e.,
  1. Assume causal structure G and functional restrictions (e.g., linear relations and independent Gaussian noise)
  2. Optimize some score (e.g., likelihood or BIC) given these restrictions
  3. Change G and compute new optimal score value
  4. Repeat this for many G and return G^{opt} with the best (optimized) score

➡ E.g., Greedy-Equivalent-Search (GES) by Chickering (2002)

Hybrid methods

- Combines constraint-based and search-and-score methods, i.e.,
  1. Constraint-based search to find skeleton
  2. Score-based approach to orient edges

➡ E.g., Max-Min Hill-Climbing (MMHC) by Tsamardinos et al. (2006)

References

Literature

- Pearl, J. (2009). *Causal inference in statistics: An overview*. Statistics Surveys.
- Pearl, J. (2009). *Causality: Models, Reasoning, and Inference*. Cambridge University Press.
- Spirtes et al. (2000). *Causation, Prediction, and Search*. The MIT Press.
- Kalisch et al. (2007). *Estimating high-dimensional directed acyclic graphs with the PC-algorithm*. Journal of Machine Learning Research.
- Colombo et al. (2014). *Order-independent constraint-based causal structure learning*. The Journal of Machine Learning Research.
- Le et al. (2016). *A fast PC algorithm for high dimensional causal discovery with multi-core PCs*. IEEE/ACM transactions on computational biology and bioinformatics.
- Kalisch et al. (2014). *Causal structure learning and inference: a selective review*. *Quality Technology & Quantitative Management*

References

Implementations

R

- Kalisch et al. (2017), R Package 'pcalg'.
- Le et al. (2015), R Package 'ParallelPC'.
- Scutari (2007), Learning Bayesian Networks with the bnlearn R Package.

Python

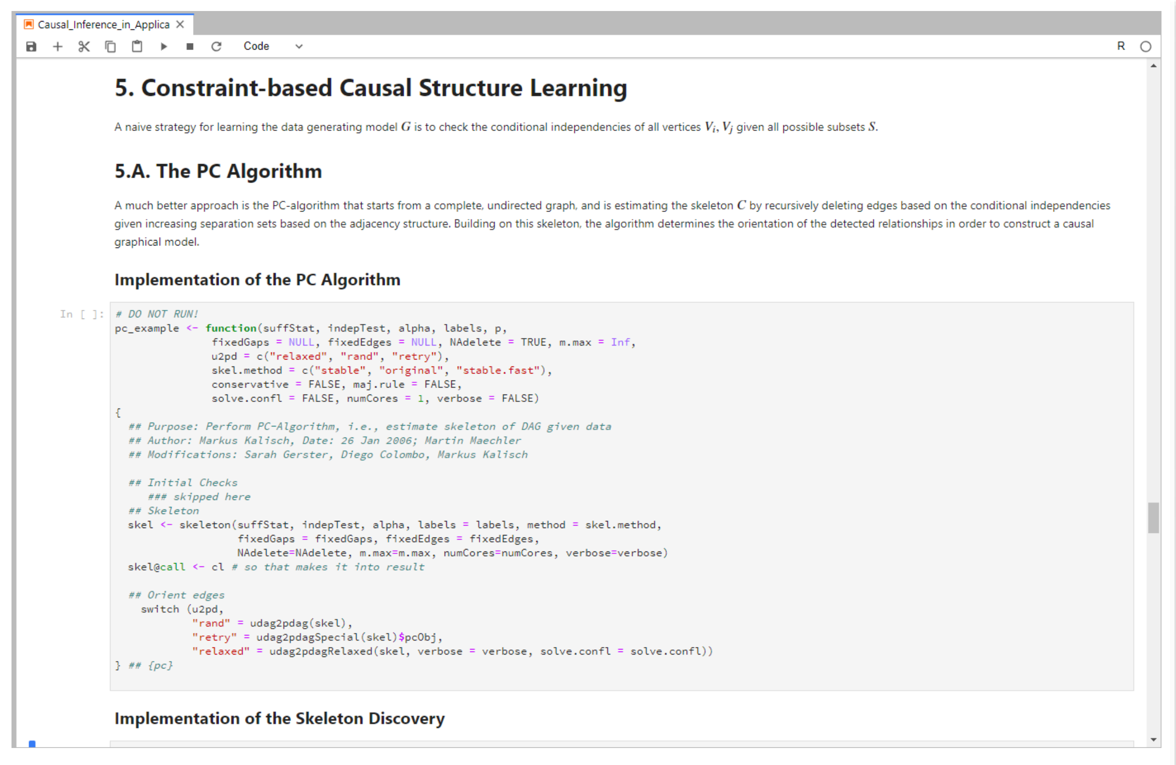
- Kobayashi (2015), CPDAG Estimation using PC-Algorithm.
  (Note: Unstable version of the PC Algorithm)

Other

- Carneggie Mellon University, The Tetrad Project

Causal Inference in Application

Causal Inference in Application
Cooling House Example



5. Constraint-based Causal Structure Learning

A naive strategy for learning the data generating model G is to check the conditional independencies of all vertices V_i, V_j given all possible subsets S.

5.A. The PC Algorithm

A much better approach is the PC-algorithm that starts from a complete, undirected graph, and is estimating the skeleton C by recursively deleting edges based on the conditional independencies given increasing separation sets based on the adjacency structure. Building on this skeleton, the algorithm determines the orientation of the detected relationships in order to construct a causal graphical model.

Implementation of the PC Algorithm

In [ ]:

```r
# DO NOT RUN!
pc_example <- function(suffStat, indepTest, alpha, labels, p,
               fixedGaps = NULL, fixedEdges = NULL, NAdelete = TRUE, m.max = Inf,
               u2pd = c("relaxed", "rand", "retry"),
               skel.method = c("stable", "original", "stable.fast"),
               conservative = FALSE, maj.rule = FALSE,
               solve.confl = FALSE, numCores = 1, verbose = FALSE)
{
  ## Purpose: Perform PC-Algorithm, i.e., estimate skeleton of DAG given data
  ## Author: Markus Kalisch, Date: 26 Jan 2006; Martin Maechler
  ## Modifications: Sarah Gerster, Diego Colombo, Markus Kalisch

  ## Initial Checks
     ### skipped here
  ## Skeleton
  skel <- skeleton(suffStat, indepTest, alpha, labels = labels, method = skel.method,
                   fixedGaps = fixedGaps, fixedEdges = fixedEdges,
                   NAdelete=NAdelete, m.max=m.max, numCores=numCores, verbose=verbose)
  skel@call <- cl # so that makes it into result

  ## Orient edges
    switch (u2pd,
            "rand" = udag2pdag(skel),
            "retry" = udag2pdagSpecial(skel)$pcObj,
            "relaxed" = udag2pdagRelaxed(skel, verbose = verbose, solve.confl = solve.confl))
} ## {pc}
```

Implementation of the Skeleton Discovery

Causal Inference
Theory and Applications in Enterprise Computing
Uflacker, Huegle, Schmidt

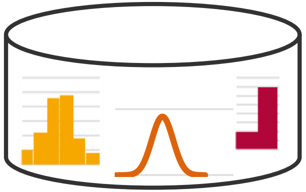
Outlook: Group Work on Research Topics

Outlook: Group Work on Research Topics

Overview on Topics

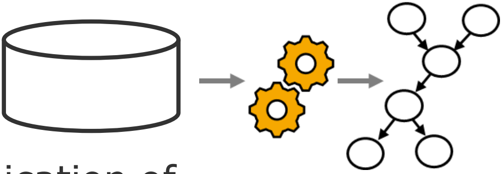
- ***Data, Distributions, Independence***

  Work on topics in the application of learnt techniques beyond the examples given in this lecture (e.g., heterogeneous data distributions)



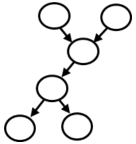
- ***Causal Structure-Learning***

  Work on topics in the context of performance improvements of causal structure learning algorithms (e.g., hardware acceleration)



- ***Applications Scenarios***

  Work on challenges and opportunities in the application of causal inference techniques on real-world data (e.g., industrial manufacturing)



Outlook: Group Work on Research Topics
Topic Application

- ***How to work on a topic?***
  1. Understand theoretic basis and your selected topic
  2. Work on implementation
  3. Present results
  4. Write scientific report in a review process

- ***How to apply for a topic?***
  - Build groups of around three students
  - Send prioritized list of top 3 topics to Johannes Huegle until:
    Fri April 26, 11.59 PM
  - Topic Assignments:
    Tue April 30, 9:00 AM

Thank you
for your attention!