Lower Bounds for the Modular Communication Complexity of Various Graph Accessibility Problems *

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Abstract. We investigate the modular communication complexity of the graph accessibility problem GAP and its modular counting versions MOD_k–GAP, $k \geq 2$. Due to arguments concerning variation ranks and certain projection reductions, we prove that, for any partition of the input variables and for any moduls k and m, GAP and MOD_k–GAP have MOD_m–communication complexity $\Omega(n)$, where n denotes the number of nodes of the graphs under consideration.

Topics: Computational Complexity, Modular Communication Protocols, Variation ranks, Projection Reductions, *GAP*, *MOD_k-GAP*

Introduction

The *graph accessibility problem* GAP $= (\mathrm{GAP}_n)_{n\in\mathbb{N}}$ consists in the decision whether there is a path in a given directed, acyclic n-node graph $G = (V, E)$, $V = \{1, \ldots, n\}$ and $E \subseteq V \times V$, that leads from vertex 1 to vertex n. As usual, let G be given by its adjacency matrix $G = (a_{ij})_{1\leq i,j\leq n, i\neq j}$ with

$$a_{ij} = a(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E\,; \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{GAP}_n : \{0,1\}^{n^2} \longrightarrow \{0,1\}$, is defined by

$$(a_{ij}) \longrightarrow \begin{cases} 1 & \text{if there is a path in the graph described by } (a_{ij}) \text{ from 1 to } n; \\ 0 & \text{otherwise.} \end{cases}$$

The major property of GAP is the following one.

Theorem 1. GAP *is complete for the complexity class* NL *of languages acceptable by nondeterministic logarithmic space-bounded Turing machines via logspace reductions (see [11]), via projection translations (see [6]), and via p-projection reductions for nonuniform* NL *(see [8]).* □

* Proc. LATIN'95, LNCS 911, 427–435.

Soon it was realized (see, e.g.,[9]) that certain modified GAPs, denoted by MOD_k–GAP, $k \geq 2$, have similar properties for the complexity classes MOD_k–L, defined by logarithmic space–bounded Turing machines equipped with the counting acceptation mode MOD_k. Here, an input is accepted, if and only if the number of accepting computations is *not* congruent 0 modulo k.

MOD_k–$\mathrm{GAP}_n : \{0,1\}^{n^2} \longrightarrow \{0,1\}$, is defined by

$$(a_{ij}) \longrightarrow \begin{cases} 1 & \text{the number of paths in } (a_{ij}) \text{ from 1 to } n \text{ is not divisible by } k; \\ 0 & \text{otherwise.} \end{cases}$$

A generalization of Theorem 1 yields the following theorem which is true for the various reduction notions. (For a proof, e.g. of the p-projection completeness, we refer to [10].)

Theorem 2. MOD_k–GAP *is complete for* MOD_k–L, $k \geq 2$. □

From Theorems 1 and 2 it becomes clear why it is an important goal in complexity theory to characterize the complexity of graph accessibility problems. In [13], Yao started the study of the communication complexity of graph problems. In [5] the deterministic communciation complexity of *connectivity* and *s-t-connectivity* for undirected graphs was investigated. The modular communication complexity was introduced in [3] and first systematically studied in [4]. Its investigation is of great interest e.g. for the investigation of problems computable by constant depth, polynomial size circuits with MOD_k–gates [14] and [7], or by bounded alternating branching program [10]. In the following, we investigate the *modular communication complexity* of GAP and MOD_k–GAP, $k \in \mathbb{N}$.

Let a graph $G = (V, E)$ be given, in arbitrarily distributed form, to two nondeterministic processors P_1 and P_2 with unbounded computational power. In order to solve GAP or MOD_k–GAP, both processors have to communicate via a common communication tape. The computation of the whole structure, which is called a *communication protocol* or simply a *protocol*, is going on in *rounds*. Starting with P_1, the processors write alternatingly bits on the communication tape. These bits depend on the input available to the processor which is to move and on the bits already written on the communication tape before. We assume without loss of generality that in each round exactly one bit is written on the communication tape and that all (nondeterministic) computations of a protocol are of equal length, say L. If the last bit written on the communication tape is "1" or "0", the particular computation is called *accepting* or *rejecting*, respectively. (Since we shall assume the processors to be nondeterministic, this last bit need not to coincide with the output of the protocol.) So co-operative computations can be thought of as to be Boolean strings. The length of the string is the *communication complexity* of the computation. (For more reading on communciation complexity we refer, e.g., to [1], [2], [3], [4]). Since our processors are nondeterministic we have to define the *output* of a protocol by means of a certain *acception mode*. In the following we consider the *modular* or, more exactly, the MOD_m*–acceptance mode* in which the protocol accepts an input, if the number of accepting computations is not equal to 0 modulo m.

Concerning the modular communciation complexity of graph accessibility problems up to now it was merely known that, for primes p, the MOD_p–communciation complexity of GAP_n as well as MOD_p–GAP_n is at least $\Omega(n)$ [10]. In the following we continue and complete these investigations by showing that this lower bound remains true

– for each MOD_m–acceptance mode, and
– for each MOD_k–GAP,

where $m,\ k \geq 2$ are arbitrary composite integers. We show:

Theorem 3. Let m and k be arbitrary numbers, $m,\ k > 1$. Then it holds

1. $MOD_m\text{–}Comm(\mathrm{GAP}_n) = \Omega(n)$.
2. $MOD_m\text{–}Comm(\mathrm{MOD}_k\text{–}\mathrm{GAP}_n) = \Omega(n)$.

In order to prove this theorem we adopt the technique of variation ranks introduced in [7], prove some lower bounds for the *sequence equality problem*, and reduce these lower bounds via certain projection reductions to the graph accessibility problems under consideration.

1 The Computational Model

In order to be able to receive our results we need precise formal definitions of the considered computational model. Let $f : S_1 \times S_2 \to \{0,1\}$ be given in distributed form. A *protocol of length* L consisting of two processors $\mathcal{P}_1$ and $\mathcal{P}_2$ which access inputs of S_1 and S_2, respectively, can be described by two functions

$$\Phi_i :\ S_i \times \{0,1\}^{\leq L} \to \{0,1\},\quad i = 1,2.$$

The interpretation is as follows. Let $\gamma = \gamma_1 \ldots \gamma_j,\ \gamma_k \in \{0,1\}$. If $\Phi_i(s_i,\gamma) = 1$, and if $|\gamma| - i$ is even, then the corresponding processor $\mathcal{P}_i$ is able to write γ_j on the communication tape provided that it has read $\gamma_1 \ldots \gamma_{j-1}$ on the communication tape and that it has s_i as input. If, however, $\Phi_i(s_i,\gamma) = 0$, then $\mathcal{P}_i$ is not able to write γ_j.

The work of a protocol P of length L can be described in terms of a $\#S_1 \times \#S_2$–matrix Acc^P giving as entry with index $(s_1,s_2) \in S_1 \times S_2$ the number of accepting computations of the protocol P on input (s_1,s_2). In order to make this approach unique, we agree that $\Phi_i(s_i,\gamma) = 1$, if $|\gamma| - i$ is odd, for $i = 1,2$.

$$Acc^P_{s_1,s_2} \stackrel{def}{=} \sum_{\gamma_1 \cdots \gamma_L \in \{0,1\}^L, \gamma_L = 1} \prod_{j=1}^{L} \Phi_{(1+(j+1)\ mod\ 2)}(s_{(1+(j+1)\ mod\ 2)}, \gamma_1 \ldots \gamma_j) \tag{1}$$

An equivalent definition of the above matrix immediately provides a lower bound on the length of the protocol in terms of its rank. Let $\gamma \in \{0,1\}^L$ be a computation. Define

$$\chi_i^P(s_i,\gamma) \stackrel{def}{=} \prod_{\gamma' \in \{0,1\}^{\leq L},\ \gamma' \preceq \gamma} \Phi_i(s_i,\gamma'), \tag{2}$$

for $i = 1, 2$. From equation (1) one directly gets

$$Acc^P_{s_1,s_2} = \sum_{\gamma \in \{0,1\}^L,\ \gamma_L = 1} \chi^P_1(s_1, \gamma) \cdot \chi^P_2(s_2, \gamma) \tag{3}$$

Equation 3 immediately yields

Lemma 4. *Let R be any semiring. Let P be a protocol of the length L on the input set $S_1 \times S_2$, $\#S_1 = \#S_2 = N$, and let Acc^P be the $N \times N$–matrix defined in equation 3. Then*

$$2^{L-1} \geq \text{rank}_R(Acc^P). \quad \square$$

Definition 5. Let m be any number, $m > 1$. A protocol P equipped with the *modular* MOD_m*–acceptance mode accepts* an input (s_1, s_2) iff $Acc^P_{s_1,s_2} \not\equiv 0\ (mod\ m)$. Otherwise P *rejects* (s_1, s_2). The function computed by P is denoted by $\text{Comp}(P, \text{MOD}_m)$. The MOD_m*-communication complexity* of a function $f : S_1 \times S_2 \longrightarrow \{0, 1\}$ is the minimal length of a protocol P with Comp(P,MOD_m)$= f$.

Lemma 6. *If $s|t$, then* MOD_t-Comm$(f) \leq log_2(\frac{t}{s}) +$ MOD_s-Comm(f)*, for each function f.*

Proof. Clearly, $t|\frac{u \cdot t}{s}$, if and only if, $s|u$. Let P be the MOD_s–protocol for f. We consider the following protocol P'.

First, processor $\mathcal{P}_1$ chooses nondeterministically an index l, $1 \leq l \leq \frac{t}{s}$ and sheds l.

Second, $\mathcal{P}'_2$ and $\mathcal{P}'_1$ proceed in the same way as the processors $\mathcal{P}_1$ and $\mathcal{P}_2$ of the MOD_s–protocol P do. We get $Acc^{P'}_{ij} = \frac{t}{s} \cdot Acc^P_{ij}$. Consequently, $Acc^{P'}_{ij} \equiv 0\ (mod\ t)$ iff $Acc^{P'}_{ij} \equiv 0\ (mod\ s)$. If L is the length of protocol P, then $log_2(\frac{t}{s}) + L$ is the length of protocol P'. $\square$

Investigating communication complexity, the appropriate type of reduction is that of rectangular reductions which are defined as follows: Let $F = (F_{2n} : \Sigma^n \times \Sigma^n \longrightarrow \{0, 1\})_{n \in \mathbb{N}}$ and $G = (G_{2n} : \Gamma^n \times \Gamma^n \longrightarrow \{0, 1\})_{n \in \mathbb{N}}$ be two decision problems. F is *rectangularly reducible* to G with respect to q (denoted by $F \leq^q_{rec} G$), where $q : \mathbb{N} \longrightarrow \mathbb{N}$ is a nondecreasing function, if, for each n, there are two transformations $l_n, r_n : \Sigma^n \longrightarrow \Gamma^{q(n)}$ such that for all $\mathbf{x}, \mathbf{y} \in \Sigma^n$ $F_{2n}(\mathbf{x}, \mathbf{y}) = G_{2q(n)}(l_n(\mathbf{x}), r_n(\mathbf{y}))$.

Rectangular reductions can be used for proving lower bounds on the MOD_m–communication complexity in the following way: Let $q : \mathbb{N} \longrightarrow \mathbb{N}$ be an unbounded nondecreasing function. Then we define $q^{(-1)}$ by $q^{(-1)}(i) = \max\{j |\ q(j) \leq i\}$. Standard arguments yield

Lemma 7. *Assume there are given two sequences of functions $F = (F_{2n} : \Sigma^n \times \Sigma^n \longrightarrow \{0, 1\})_{n \in \mathbb{N}}$ and $G = (G_{2n} : \Gamma^n \times \Gamma^n \longrightarrow \{0, 1\})_{n \in \mathbb{N}}$. If $c(n) \leq MOD_m$-*Comm(F) *and $F \leq^q_{rec} G$, then*

$$c \circ q^{(-1)}(n) \leq MOD_m\text{-Comm}(G). \quad \square$$

One efficient way to get rectangular reductions is to work with projection reductions [12] which are defined as follows.

Definition 8. Let $F = (F_n : \{0,1\}^n \rightarrow \{0,1\})_{n \in \mathbb{N}}$ and $G = (G_n : \{0,1\}^n \rightarrow \{0,1\})_{n \in \mathbb{N}}$. The mapping $\pi_n : \{y_1, \ldots, y_m\} \rightarrow \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n, 0, 1\}$ is called a *projection reduction* from F_n to G_m if $F_n(x_1, \ldots, x_n) = G_m(\pi(y_1), \ldots, \pi(y_m))$. If F_n and G_m are given in distributed form,

$$F_{2n} : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\} \text{ and } G_{2m} : \{0,1\}^m \times \{0,1\}^m \rightarrow \{0,1\}$$

then a projection reduction π_n is said to *respect the distribution* of the variables if

$$\pi_n^{-1}\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\} \subseteq \{y_1, \ldots, y_m\}$$

and

$$\pi_n^{-1}\{x_{n+1}, \ldots, x_{2n}, \neg x_{n+1}, \ldots, \neg x_{2n}\} \subseteq \{y_{m+1}, \ldots, y_{2m}\}.$$

A sequence $\pi = (\pi_n)_n \in \mathbb{N}$ of reduction projections π_n is called a *$p(n)$–projection reduction* and we write $F \leq_\pi^p G$ if $p(n)$ is a nondecreasing function with $m \leq p(n)$.

From Lemma 7 we immediately get

Lemma 9. *Assume that we are given two sequences of functions $F = (F_{2n} : (\{0,1\})^n \times (\{0,1\})^n \rightarrow \{0,1\})_{n \in \mathbb{N}}$ and $G = (G_{2m} : (\{0,1\})^m \times (\{0,1\})^m \rightarrow \{0,1\})_{m \in \mathbb{N}}$ with $F \leq_\pi^p G$, where p is increasing and $\pi = (\pi_n)_{n \in \mathbb{N}}$ is a sequence of projection reductions that respects the distribution of the variables. If $c(n) \leq MOD_m$-*Comm(F)*, then $c \circ q^{(-1)}(n) \leq MOD_m$-*Comm$(G)$. $\square$

2 Rank Arguments for Lower Bounds

Throughout this section let f denote a function $f : S_1 \times S_2 \rightarrow \{0,1\}$, $N = \#S_1 = \#S_2$, and let M^f denote the communication matrix, where $M^f_{i,j} = f(i,j)$, for $i, j = 1, \ldots, N$.

We start with adopting the concept of variation ranks of communication matrices introduced in [7].

Definition 10. Two $N \times N$–matrices A and B with coefficients from $\mathbb{Z}$are said to be *MOD_m–equivalent*, $m > 1$, if, for all indices i, j,

$$a_{ij} \equiv 0 \ (mod\ m) \quad \text{iff} \quad b_{ij} \equiv 0 \ (mod\ m).$$

The *variation rank* var-rank$_{\mathbb{Z}/m\mathbb{Z}}(A \ mod\ m)$ is the minimum over all numbers rank$_{\mathbb{Z}/m\mathbb{Z}}(B \ mod\ m)$, where B is MOD_m–equivalent to A.

A 0–1 matrix is interpreted as an R–matrix, where R denotes an arbitrary semiring, in the canonical way. As usual, *the R–rank of a $m \times n$–matrix A over R*, which we denote by rank$_R A$, is defined to be the minimal number s such that $A = B \cdot C$, where B is a $r \times s$–matrix and C is a $s \times t$–matrix over R. A straightforward calculation yields the next lemma.

Lemma 11. *Let A be an integer matrix. Then*

$$\mathrm{rank}_{\mathbb{Z}/m\mathbb{Z}}(A \; mod \; m) = \min\{\mathrm{rank}_{\mathbb{Z}} B \mid B \text{ is } \mathrm{MOD}_m\text{–equivalent to } A\}. \quad \square$$

Due to Lemmas 1 and 5 the modular communication complexity can be characterized in terms of variation ranks which was first proved in [4].

Proposition 12. *It holds* $\log_2 \left(\text{var-rank}_{\mathbb{Z}/m\mathbb{Z}}(M^f)\right) \leq \mathrm{MOD}_m\text{-Comm}(f)$. $\square$

By means of Proposition 12, one can derive lower bounds on the modular communication complexity by computing lower bounds on the variation rank of certain problems. In order to do this we consider the the well-know *sequence equality function SEQ_{2n}* which is defined by

$$\mathrm{SEQ}_{2n}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \bigwedge_{i=1}^{n} (x_i = y_i).$$

(Here $S_1 = S_2 = \{0,1\}^n$.) The great importance of this function in communication complexity investigations is based on the trivial fact that the communication matrix $M^{SEQ_{2n}}$ of SEQ_{2n} equals the $2^n \times 2^n$ identity matrix I_{2^n}.

Proposition 13. *For m arbitrary, it holds* $\mathrm{MOD}_m\text{-Comm}(\mathrm{SEQ}_{2n}) = \Omega(n)$.

Proof. The claim follows from Proposition 12 and the following Lemma 14 which is an improvement of a corresponding lemma in [7] where the case of primes is considered.

Lemma 14. *Let I_N denote the identity $N \times N$–matrix. Let $m = p_1^{l_1} \cdot \ldots \cdot p_r^{l_r}$ be a natural number which is given by its primary decomposition. Then*

$$\text{var-rank}_{\mathbb{Z}/m\mathbb{Z}}(I_N) = \lceil N/r \rceil.$$

Proof. First we prove that $\lceil N/r \rceil$ is a lower bound.
Let B be an integer matrix such that B is MOD_m–equivalent to I_N and $\text{var-rank}_{\mathbb{Z}/m\mathbb{Z}}(I_N) = \mathrm{rank}_{\mathbb{Z}} B$, which exists by Lemma 11. By definition we have, for all i, $b_{ii} \not\equiv 0 \; (mod \; m)$, and, for all $j \neq i$, $a_{ij} \equiv 0 \; (mod \; m)$. For all $i \in \{1, \ldots, n\}$ there is a $k \in \{1, \ldots, r\}$ such that $b_{ii} \not\equiv 0 \; (mod \; p^{l_k})$. We conclude that there is a primary component $p_k^{l_k}$ of m, which we denote for simplicity by p^l, a set of indices

$$\mathcal{I} \subseteq \{1, 2, \ldots, N\}, \quad \#\mathcal{I} \geq N' := \lceil N/r \rceil,$$

and, for all $i \in \mathcal{I}$, natural numbers $\nu_i \in \{1, \ldots, l_i\}$, such that

$$\begin{aligned} b_{ii} &\equiv 0 \; (mod \; p^{l-\nu_i}), \\ b_{ii} &\not\equiv 0 \; (mod \; p^{l-\nu_i+1}), \\ b_{ij} &\equiv 0 \; (mod \; p^{l}), \end{aligned}$$

for all $j \in \mathcal{I}$, $j \neq i$. After deleting all rows and columns of B whose indices do not belong to $\mathcal{I}$, we get an integer $N' \times N'$–matrix C. It is sufficient to show that $det\, C \neq 0$. It is easy to see that

$$c_{1,1} \cdot \ldots \cdot c_{N',N'} \not\equiv 0 \ (mod\ p^{N' \cdot l+1-\sum_{i=1}^{N'} \nu_i}),$$

but

$$c_{1,\sigma(1)} \cdot \ldots \cdot c_{N',\sigma(N')} \equiv 0 \ (mod\ p^{N' \cdot l+1-\sum_{i=1}^{N'} \nu_i}),$$

for all permutations σ of the set $\{1, \ldots, N'\}$ different from the identity permutation. Consequently,

$$det\, C \equiv c_{1,1} \cdot \ldots \cdot c_{N',N'} \not\equiv 0 \ (mod\ p^{N' \cdot l+1-\sum_{i=1}^{N'} \nu_i}).$$

Second, we prove that $\lceil N/r \rceil$ is an upper bound.
Let $f_i = p_i^{-l_i} \prod_{\mu=1}^{r} p_\mu^{l_\mu}$, $F_j = (f_1, \ldots, f_j)$, and $A_j = F_j^T \cdot F_j$ for $i, j = 1 \in \{1, \ldots, r\}$. A_0 is defined to be the unique 0×0–matrix, which, of course, has rank 0. Clearly, $(A_j \ mod\ m)$ is a $j \times j$–diagonal matrix of $\mathbb{Z}/m\mathbb{Z}$–rank 1, for each $j \in \{1, \ldots, r\}$. Define matrix A to be the following direct sum of matrices.

$$A \stackrel{def}{=} A_r \oplus \overset{\lfloor N/r \rfloor}{\ldots\ldots} \oplus A_r \oplus A_{r'},$$

where $r' \equiv N \ (mod\ r)$, and $r' \in \{0, \ldots, r-1\}$. It follows that $(A \ mod\ m)$ is a diagonal $N \times N$–matrix which is MOD$_m$–equivalent to I_N , and has $\mathrm{rank}_{\mathbb{Z}/m\mathbb{Z}}(A \ mod\ m) \leq \lfloor N/r \rfloor + 1$. $\square$

3 Graph Accessibility Problems

Now we use the lower bound of Proposition 13 in order to derive lower bounds on the MOD_m–communication complexity ($m > 1$) of the graph accessiblity problems GAP and MOD_k-GAP ($k > 1$) given in distributed from. We do this by reducing SEQ via a projection reduction π that respects the given input distribution to GAP and MOD_k-GAP, respectively. Speaking about projection reductions in the sequel, we always mean such ones that respect the preassigned partition. We visualize the graph, which is the image under π in such a way that the edges which are not constant are drawn as thin lines and are labelled with literals. The meaning is that such an edge belongs to the graph, if and only if, this literal is true. All other edges are drawn as thick lines.

We start with an easy graph theoretical lemma which shows that any partition of the complete graph provides "enough space" to define a projection reduction that respects a given partition.

Lemma 15. *Let $E_1 \cup E_2$ be any partition of the set of all edges $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into two sets of equal size $\frac{n(n-1)}{2}$. Then there are subsets $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$ such that*

(i) $\#E'_1 = \#E'_2 \geq \lfloor n/8 - 1 \rfloor$,
(ii) *the edges from* $E'_1 \cup E'_2$ *are pairwise vertex–disjoint, and*
(iii) *neither vertex* 1 *nor vertex* n *is incident with any edge from* $E'_1 \cup E'_2$.

Proof. We observe that if we select two vertex–disjoint edges from E_1 and E_2 we have to remove at most $4n-2$ edges from E_1 and from E_2, which are not vertex-disjoint with the chosen edges. $\square$

Proposition 16. SEQ $=$ *(*SEQ_{2n}*) is reducible to* GAP *=* (GAP_n) *and to* MOD_k–GAP $= (\mathrm{MOD}_k\text{–}\mathrm{GAP}_n)$*,* $k \geq 2$*, given in distributed form via a* $O(n^2)$*–projection reduction which respects the partition of the variables.*

Proof. Let GAP $= (\mathrm{GAP}_n)_{n \in \mathbb{N}}$ or MOD_k–GAP $= (\mathrm{MOD}_k\text{–}\mathrm{GAP}n)_{n \in \mathbb{N}}$, $k \geq 2$, be given in distributed form by $\{x_{i,j} \,|\, (i,j) \in E_1\}$ and $\{y_{i,j} \,|\, (i,j) \in E_2\}$, $\#E_1 = \#E_2 = n(n-1)/2$. Let E'_1 and E'_2 be subsets of E_1 and E_2 fulfilling the conditions of Lemma 15 with $\#E'_1 = \#E_2 = 2r$, $r = \Theta(n)$. Let $E'_1 = \{(a_{\nu_1}, b_{\nu_1}),\ (a_{\nu_2}, b_{\nu_2}) \mid \nu = 1, \ldots, r\}$, and $E'_2 = \{(c_{\nu_1}, d_{\nu_1}),\ (c_{\nu_2}, d_{\nu_2}) \mid \nu = 1, \ldots, r\}$.

Now we consider an input $(t_1, \ldots, t_r, u_1, \ldots, u_r)$ of SEQ_{2r} and define a projection reduction

$$\pi_{2r} : \{x_{ij}, y_{ij} \mid i,j = 1, \ldots, n\} \rightarrow \{0, 1, t_\nu, u_\nu, \neg t_\nu, \neg u_\nu \ \mid\ \nu = 1, \ldots, r\}.$$

as illustrated in Figure 1 where the image of π_{2r} is shown. The idea is to assign to the variables $x_{i,j}$, $(i,j) = (a_{\nu_1}, b_{\nu_1}) \in E'_1$, and $y_{i,j}$, $(i,j) = (c_{\nu_1}, d_{\nu_1}) \in E'_2$, the literals t_ν and u_ν, respectively, and to the variables $x_{i,j}$, $(i,j) = (a_{\nu_2}, b_{\nu_2}) \in E'_1$, and $y_{i,j}$, $(i,j) = (c_{\nu_2}, d_{\nu_2}) \in E'_2$, the literals $\neg t_\nu$ and $\neg u_\nu$, respectively, what indeed can be done in the cited manner due to Lemma 15. The remaining variables are assigned to 0 with exeption of the variables corresponding to the thick line connections in the picture.

It is clear, that this graph has at most one path from 1 to n which exists if and only if

$$\mathrm{SEQ}_{2n}(t_1, \ldots, t_r, u_1, \ldots, u_r) = 1. \ \square$$

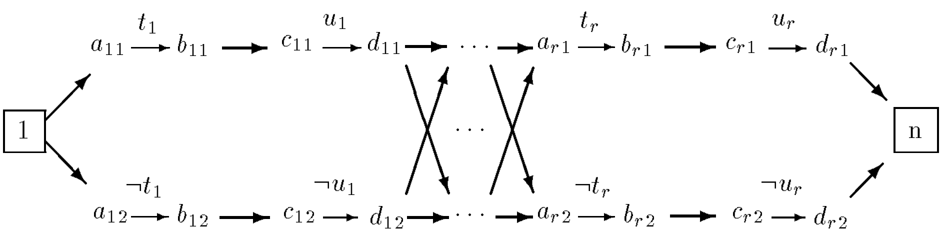

Figure 1 *Construction of the graph that is the image of* $(t_1, \ldots, t_r, u_1, \ldots, u_r)$ *under* π.

Altogether we obtain now teh above mentioned main result. of this paper:

Theorem 17. Let m and k be arbitrary numbers, $m,\ k \geq 2$. Then it holds

1. $MOD_m\text{-}Comm(\mathrm{GAP}_n) = \Omega(n)$.
2. $MOD_m\text{-}Comm(\mathrm{MOD}_k\text{-}\mathrm{GAP}_n) = \Omega(n)$. □

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