Fixed Parameter Multi-Objective Evolutionary Algorithms for the W-Separator Problem

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ABSTRACT

Parameterized analysis provides powerful mechanisms for obtaining fine-grained insights into different types of algorithms. In this work, we combine this field with evolutionary algorithms and provide parameterized complexity analysis of evolutionary multi-objective algorithms for the W-separator problem, which is a natural generalization of the vertex cover problem. The goal is to remove the minimum number of vertices such that each connected component in the resulting graph has at most W vertices. We provide different multi-objective formulations involving two or three objectives that provably lead to fixed-parameter evolutionary algorithms with respect to the value of an optimal solution OPT and W. Of particular interest are kernelizations and the reducible structures used for them. We show that in expectation the algorithms make incremental progress in finding such structures and beyond. The current best known kernelization of the W-separator uses linear programming methods and requires a non-trivial post-process to extract the reducible structures. We provide additional structural features to show that evolutionary algorithms with appropriate objectives are also capable of extracting them. Our results show that evolutionary algorithms with different objectives guide the search and admit fixed parameterized runtimes to solve or approximate (even arbitrarily close) the W-separator problem.

KEYWORDS

Evolutionary Algorithms, Parameterized Complexity, Runtime Analysis

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1 INTRODUCTION

Parameterized analysis of algorithms [5] provides a way of understanding the working behaviour of algorithms via their dependence on important structural parameters for NP-hard problems. This technique of fine-grained analysis allows for insights into which parameters make a problem hard. When analyzing heuristic search methods such as evolutionary algorithms, a parameterized runtime analysis allows for runtime bounds not just dependent on the given input size but also in terms of parameters that measure the difficulty of the problem. This is particularly helpful for understanding heuristic search methods which are usually hard to analyze in a rigorous way.

The area of runtime analysis has contributed to the theoretical understanding of evolutionary algorithms and other bio-inspired algorithms from various perspectives [4, 8, 16]. Parameterized analysis of evolutionary algorithms has been carried out for several important combinatorial optimization problems (see [15] for an overview). The first analysis was for the classical vertex cover problem [11] which is the prime problem in the area of parameterized complexity. Following that, problems such as the maximum leaf spanning problem [10], the Euclidean traveling salesperson problem [20] and parameterized settings of makespan scheduling [19] were considered. More recently, both the closest string problem [18] and jump and repair operators have been analyzed in the parameterized setting [2]. A crucial aspect of the parameterized analysis of evolutionary algorithms (and algorithms in general) is the ability of the considered approaches to obtain a kernelization for the respective problems. A kernel here refers to a smaller sub-problem whose size is polynomially bounded in the size of the given parameter(s). As the size is bounded, brute-force methods or random sampling can then be applied to obtain an optimal solution.

A small subset of vertices that disconnect a graph is usually called a vertex separator. In terms of successful divide-and-conquer or parallel processing strategies, such separators are one of the most powerful tools for developing efficient graph algorithms. This generality and its broad applicability have made the study of separators a rich and active field of research, see for example the book by Rosenberg and Heath [17], or the line of research initiated by the seminal work of Lipton and Tarjan [14] on separators in planar graphs. Numerous different types of separator structures have emerged over the past couple of decades. In this paper, we address the problem of decomposing a graph into small pieces - with respect to a parameter W - by removing the smallest possible set of vertices. More formally, given a graph $G = (V, E)$ and a parameter $W \in \mathbb{N}$, the goal is to remove the minimum number of vertices such that each connected component in the resulting graph has at most W vertices. The problem is called the *W-separator problem* — also known in the literature as the *component order connectivity problem* or *α-balanced separator problem*, where $\alpha \in (0, 1)$ and $W = \alpha|V|$. An equivalent view of this problem is to ask for the minimum number of vertices required to cover or hit every connected subgraph of size $W + 1$. In particular, $W = 1$ corresponds to covering all edges, showing that the W-separator problem is a natural generalization of the vertex cover problem.

In this paper, we generalize the results obtained in [11] for the vertex cover problem to the more general W-separator problem. Precisely, we study multi-objective evolutionary algorithms for the W-separator problem and show that in expectation they admit fixed parameter runtimes with respect to the value of an optimal solution OPT and W. It is unlikely that such runtimes can be achieved by considering OPT or W alone. Indeed, $W = 1$ corresponds to a hard problem, which shows that W (alone) is not a suitable parameter. For the parameter OPT, the problem is $W[1]$-hard even when restricted to split graphs [6]. These lower bounds lead to the study of parameterization by W + OPT. The best known algorithm with respect to these parameters finds an optimal solution in time $n^{O(1)} \cdot 2^{O(\log(W)\cdot \mathrm{OPT})}$ [6]. Unless the exponential time hypothesis fails, the authors prove that this running time is tight up to constant factors, i.e., there is no algorithm that solves the problem in time $n^{O(1)} \cdot 2^{o(\mathrm{OPT}\cdot\log(W))}$. For kernelizations with respect to the parameters OPT and W, the best known polynomial algorithm achieves a kernel of size $3W \cdot$ OPT [3]. A kernel of size $2W \cdot$ OPT is provided in [12] in a runtime of $n^{O(1)} \cdot 2^{O(W)}$ by using linear programming methods (the runtime is not specified in the paper, but can be realized as already mentioned in [7] Section 6.4.2). In particular, for the vertex cover problem (i.e., $W = 1$), they obtain a $2 \cdot$ OPT size-kernel implying that they also obtain 2-approximation. That is, under the assumption that the unique games conjecture is true, $2W \cdot$ OPT is the best kernel we can hope for [9]. Finally, the best known approximation algorithm also uses linear programming methods and has a multiplicative gap guarantee of $O(\log(W))$ to the optimal solution with a running time of $n^{O(1)} \cdot 2^{O(W)}$ [13]. They also showed that the superpolynomial dependence on W may be needed to achieve a polylogarithmic approximation.

Our Contribution: Of particular interest in our work are kernelizations and the reducible structures used for them. We show that in expectation the algorithms make incremental progress in finding such structures and beyond. Compared to the vertex cover problem, kernelization algorithms that are linear in OPT for the W-separator problem are more complicated (cf. [3, 12, 21]). The current best known kernelization of the W-separator uses linear programming methods and requires a non-trivial post-process to extract the reducible structures [12]. The challenge in this paper is to show that natural objectives combined with simple mutations are also capable of extracting them. To this end, we add additional structural features to the reducible structures used in [12]. Essentially, our results show that evolutionary algorithms with different objectives guide the search and admit fixed parameterized runtimes to solve or approximate (even arbitrarily close) the W-separator problem.

The different runtimes are given in this paper in terms of the number of iterations, but the tractability with respect to the considered parameters also applies when we include search point evaluations. In the following, we roughly describe the runtimes achieved with respect to the search point evaluations for exact and approximate solutions, where all results are given in expectation. We consider simple and problem-independent evolutionary algorithms in combination with three different multi-objective fitness functions. The first consists of relatively simple calculations to evaluate the search points and allows us to achieve a running time of $n^{O(1)} \cdot 2^{O(\mathrm{OPT}^2\cdot W^2)}$ to find an optimal solution. For the second and third fitness functions, stronger objectives are used in the sense of applying linear programming methods. We prove that with such evaluations the optimal solution can be found in time $n^{O(1)} \cdot 2^{O(\mathrm{OPT}\cdot W)}$. Moreover, depending on the choice of an $\varepsilon \in [0, 1)$ we obtain solutions arbitrary close to an optimal one, where the according algorithm is tractable with respect to the parameters OPT and W. As usual, the larger ε, the worse the gap guarantee, but with better running time, where $\varepsilon = 0$ corresponds to the above running time in finding an optimal solution. This result shows that we can hope for a gradual progress until an optimal solution is reached.

Our results show that in expectation evolutionary algorithms are asymptotically not far away from the problem-specific ones, where the evolved algorithms are close to the lower bounds for the W-separator problem.

Overview of the paper. The paper is organized as follows: Section 2 are the preliminaries and includes the notation, the multi-objective functions and the algorithms we work with. A runtime analysis of the considered algorithms for finding exact solutions for degree-based and LP-based fitness functions are presented in Sections 3 and 4, respectively. Finally, Section 5 is dedicated to the analysis of running times for approximations. Moreover, due to space constraints all omitted proofs can be found in the full version [1].

2 PRELIMINARIES

In the following paragraphs we present the notation, the multi-objective functions, and the algorithms we work with.

Graph Terminology. We begin with a brief introduction to the graph terminology we use in this paper. Let $G = (V, E)$ be a graph. For a subgraph $G' = (V', E')$ of G we use $V(G')$ and $E(G')$ to denote V' and E', respectively. We define the *size* of a subgraph $G' \subseteq G$ as the number of its vertices, where we denote the size of G by n. For $v \in V$ we define $N(v)$ as its neighborhood, and $d(v)$ as the degree of v. For a vertex subset $V' \subseteq V$ we define $G[V']$ as the induced subgraph of V', $G - V' := G[V \setminus V']$ and $N(V') := (\bigcup_{v\in V'} N(v)) \setminus V'$. Finally, in the context of this work, we also use directed graphs in the sense of flow networks, where we move the corresponding terminology to the appendix next to the proofs.

Parameterized Terminology. We use the standard terminology for parameterized complexity, which is also used, for example, in [5, 7]. A *parameterized problem* is a decision problem with respect to certain instance parameters. Let I be an instance of a parameterized problem with an instance parameter k, usually given as a pair (I, k). If for each pair (I, k) there exists an algorithm that solves the decision problem in time $f(k) \cdot |I|^c$, where f is a computable function and c is a constant, then the parameterized problem is *fixed-parameter tractable.* We say (I, k) is a *yes-instance* if the answer to the decision problem is positive, otherwise we say (I, k) is a *no-instance.*

Of particular interest in this work are kernelizations, which can be roughly described as formalized preprocessings. More formally, given an instance (I, k) of a parameterized problem, a polynomial algorithm is called a kernelization if it maps any (I, k) to an instance (I', k') such that (I', k') is a yes-instance if and only if (I, k) is a yes-instance, $|I'| \leq g(k)$, and $k' \leq g'(k)$ for computable functions g, g'.

The idea of parameterized complexity can be extended by combining multiple parameters. That is, if we consider an instance I with parameters $k_1, \ldots, k_m$, then we are interested in algorithms that solve the corresponding decision problem in a runtime of $f(k_1, \ldots, k_m) \cdot |I|^c$, where f is a computable function and c is a constant. We refer to runtimes that satisfy this type of form as *FPT-times.*

Problem Statement and Objectives. First we introduce the *W-separator problem.* Given is a graph $G = (V, E)$ and two positive integers k and W. The challenge is to find a vertex subset $V' \subseteq V$, such that V' has cardinality at most k and the removal of V' in G leads to a graph that contains only connected components of size at most W. The minimization problem is to find V' with the smallest cardinality, where we denote the optimal objective value by OPT. Note that we can reformulate the problem statement by demanding that V' intersects with each connected subgraph of G of size $W + 1$. In the case $W = 1$ a separator needs to cover each edge, which shows that the W-separator problem is a natural generalization of the well-known vertex cover problem.

In terms of evolutionary algorithms, a solution to the W-separator problem can be represented in a bit sequence of length n. Each vertex has value zero or one, where one stands for the vertex being part of the W-separator. Let $\{0, 1\}^n$ be our solution space. We work with multi-objective evolutionary algorithms, which evaluate each search point $X \in \{0, 1\}^n$ using a fitness function $f : \{0, 1\}^n \to \mathbb{R}^m$ with m different objectives. The goal is to minimize each of the objectives. Denote by $f^i(X)$ the i-th objective, evaluated at a search point X. For two search points X_1 and X_2, we say X_1 *weakly dominates* X_2 if $f^i(X_1) \leq f^i(X_2)$ for every $i \in [m]$, where $[m]$ is defined as the set $\{1, \ldots, m\}$. In this case, we simply write $f(X_1) \leq f(X_2)$. If additionally $f(X_1) \neq f(X_2)$, then we say that X_1 *dominates* X_2. We distinguish between Pareto-optimal search points X and vectors $f(X)$. A Pareto-optimal search point is a search point that is not even weakly dominated by any other search point, whereas a Pareto-optimal vector is not dominated by any other vector. That is, if $f(X_1)$ is a Pareto-optimal vector, then there can be a vector $X_2 \neq X_1$ with $f(X_2) = f(X_2)$, whereas if X_1 is a Pareto-optimal search point, then there is no search point $X_2 \neq X_1$ with $f(X_1) = f(X_2)$.

For some fitness functions we investigate, we use a linear program to evaluate the search points. Let $G = (V, E)$ be an instance of the W-separator problem and let $y_v \in \{0, 1\}$ be a variable for each $v \in V$. An integer program (IP) that solves the W-separator problem can be formulated as follows:

$$\min \sum_{v \in V} y_v$$
$$\sum_{v \in S} y_v \geq 1, \forall S \subseteq V \colon |S| = W + 1 \text{ and } G[S] \text{ is connected.}$$

We will consider the relaxed version of the IP by allowing fractional solutions and consider the corresponding linear program (LP). That is, instead of $y_v \in \{0, 1\}$ we have $y_v \geq 0$ for all $v \in V$. In the rest of this paper we will call it the *W-separator LP.* We define $\mathrm{LP}(G')$ for a subgraph $G' \subseteq G$ as the objective of the W-separator LP with G' as input graph. If we put every connected subgraph of size $W + 1$ as constraint in the LP formulation of the W-separator, then we end up with a running time of $n^{O(W)}$. However, as mentioned already in Fomin et. al. [7] (Section 6.4.2) finding an optimal solution for the LP can be sped up to a running time of $2^{O(W)} n^{O(1)}$. Roughly speaking, the idea is to use the ellipsoid method with separation oracles to solve the linear program, where the separation oracle uses a method called color coding that makes it tractable in W.

Next, we define few additional terms before we get to the multi-objective fitness functions. Let $X \in \{0, 1\}^n$ be a search point. For $v \in V$ we define $x_v \in \{0, 1\}$ as the corresponding value in the bit-string X. We denote by $X_1 \subseteq V$ the vertices with value one. We define $u(X)$ as the set of vertices that are in components of size at least $W + 1$ after the removal of X_1 in G. The function $u(X)$ can be interpreted as the uncovered portion of the graph with respect to the vertices X_1. The fitness functions we work with are the following:

- $f_1(X) := \left(|X_1|, |u(X)|, -\sum_{v \in X_1} d(v)\right)$,
- $f_2(X) := (|X_1|, |u(X)|, \mathrm{LP}(G[u(X)]))$,
- $f_3(X) := (|X_1|, \mathrm{LP}(G[u(X)]))$.

As the names suggest, we use *one-objective*, *uncovered-objective*, *degree-objective* and *LP-objective* to denote $|X_1|$, $|u(X)|$, $-\sum_{v \in X_1} d(v)$ and $\mathrm{LP}(G[u(X)])$ respectively. Note that the fitness f_3 is same as f_2 without the uncovered-objective. Furthermore, we use $*$ to denote that an objective can be chosen arbitrarily, for instance in $(|X_1|, *, -\sum_{v \in X_1} d(v))$ the uncovered-objective $u(X)$ is arbitrarily.

Algorithms. We proceed by presenting the algorithms that we study. All of them are based on `Global Semo` (see Algorithm 1), which maintains a population $\mathcal{P} \subseteq \{0, 1\}^n$ of n-dimensional bit strings.

We define the Algorithm `Global Semo Alt` similarly to the Algorithm `Global Semo` (see Algorithm 1) with the difference that the mutation in line 5 is exchanged by Alternative Mutation Operator (see Algorithm 2). The following two lemmas will be useful throughout the whole paper. Their proofs are similar to some appearing in [11] and due to space constraints we have moved them to the full version [1].

Algorithm 1: `Global Semo`

Choose $X \in \{0,1\}^n$ uniformly at random
$\mathcal{P} \leftarrow \{X\}$
while *stopping criterion not met* **do**
  Choose $X \in \mathcal{P}$ unformly at random
  $Y \leftarrow$ flip each bit of X independently with probability $1/n$
  If Y is not dominated by any other search point in $\mathcal{P}$, include Y into $\mathcal{P}$ and delete all other bit strings $Z \in \mathcal{P}$ which are weakly dominated by X from $\mathcal{P}$, i.e., those with $f(Y) \le f(Z)$.
end

Algorithm 2: Alternative Mutation Operator

Choose $b \in \{0,1,2\}$ uniformly at random
if $b = 2$ ***and*** $u(X) \neq \emptyset$ **then**
  $Y \leftarrow$ for $v \in u(X)$ flip each bit x_v with probability $1/2$
else if $b = 1$ ***and*** $X_1 \neq \emptyset$ **then**
  $Y \leftarrow$ for $v \in X_1$ flip each bit x_v with probability $1/2$
else
  $Y \leftarrow$ flip each bit of X independently with probability $1/n$
end

Lemma 2.1. *Let $\mathcal{P} \neq \emptyset$ be a population for the fitness functions f_1 and f_2. In the Algorithms* `Global Semo` *and* `Global Semo Alt`*, selecting a certain search point $X \in \mathcal{P}$ has probability $\Omega(1/n^2)$, and additionally flipping only one single bit in it has probability $\Omega(1/n^3)$.*

Let 0^n be the search point that contains only zeroes. Note that once 0^n is in the population it is Pareto-optimal for all fitness functions because of the one-objective.

Lemma 2.2. *Using the fitness functions f_1 or f_2, the expected number of iterations of* `Global Semo` *or* `Global Semo Alt` *until the population $\mathcal{P}$ contains the search point 0^n is $O(n^3 \log n)$.*

The following lemmas are proven analogously to Lemmas 2.1 and 2.2 by observing that the worst-case bounds on the population size decrease by a factor of n when using fitness function f_3 instead of f_1 or f_2.

Lemma 2.3. *Let $\mathcal{P} \neq \emptyset$ be a population for the fitness function f_3. In the Algorithms* `Global Semo` *and* `Global Semo Alt`*, selecting a certain search point $X \in \mathcal{P}$ has probability $\Omega(1/n)$, and additionally flipping only one single bit in it has probability $\Omega(1/n^2)$.*

Lemma 2.4. *Using the the fitness function f_3, the expected number of iterations of* `Global Semo` *or* `Global Semo Alt` *until the population $\mathcal{P}$ contains the search point 0^n is $O(n^2 \log n)$.*

3 ANALYSIS FOR DEGREE-BASED FITNESS FUNCTION

In this section we investigate the fitness f_1 on `Global Semo Alt`. We will prove that the algorithm finds an optimal W-separator in expectation in FPT-runtime with the parameters OPT and W. Recall that the parameter k in the decision variant of the W-separator asks for a W-separator of size at most k. A more general variant, known as *weighted component order connectivity problem*, was studied in [6] by Drange et al. They achieve a $O(k^2W+W^2k)$ vertex-kernel, which also holds for the W-separator problem.

Theorem 3.1 ([6], Theorem 15). *The W-separator admits a kernel with at most $kW(k+W)+k$ vertices, where k is the solution size.*

Essentially, they use the following *reduction rule*: as long as there is a vertex with degree greater than $k+W$, the vertex is included in the solution set and may be removed from the instance.

It is not difficult to see that this vertex must be included in the solution, since otherwise we would have to take more than k vertices from its neighborhood to get a feasible solution. After using this reduction rule exhaustively each vertex in the reduced instance has degree at most $k+W$. Consequently, in the reduced instance, each vertex of a W-separator is connected to at most $k+W$ connected components after its removal, where each of those components has size at most W. A simple calculation provides finally the vertex-kernel stated in Theorem 3.1.

Now, we make use of the degree-objective from f_1 to find a search point that selects those vertices which can be safely added to an optimal solution according to the reduction rule.

Lemma 3.2. *Using the fitness function f_1, the expected number of iterations of* `Global Semo Alt` *where the population $\mathcal{P}$ contains a solution X in which for all $u \in u(X)$ and for all $v \in X_1$ we have $d(u) \le OPT + W$ and $d(v) > OPT + W$ is bounded by $O(n^3(OPT + \log n))$.*

With Lemma 3.2 in hand we can upper bound the expected number of iterations that `Global Semo Alt` takes to find an optimal W-separator with respect to the fitness f_1. Note that the uncovered-objective of f_1 ensures that the algorithm `Global Semo Alt` converges to a feasible solution and that a search point X with $f_1(X) = (\mathrm{OPT}, 0, *)$ corresponds to an optimal W-separator.

Theorem 3.3. *Using the fitness function f_1, the expected number of iterations of* `Global Semo Alt` *until it finds a minimum W-separator in $G = (V,E)$ is upper bounded by $O\left(n^3(OPT + \log n) + n^2 \cdot 2^q\right)$, where $q = OPT \cdot W(OPT + W) + OPT$.*

4 ANALYSIS FOR LP-BASED FITNESS FUNCTION

In this section we investigate f_2 on `Global Semo Alt`. The main result of this section is the following theorem.

Theorem 4.1. *Let $G = (V,E)$ be an instance of the W-separator problem. Using the fitness function f_2, the expected number of iterations of* `Global Semo Alt` *until an optimal solution is sampled is upper bounded by $O(n^3(\log n + OPT) + n^2 \cdot 4^{OPT \cdot W})$.*

First we give a brief overview of a reducible structure concerning the W-separator problem associated with the objectives in the fitness function f_2. The structure we will use is commonly known as crown decomposition. Roughly speaking, it is a division of the set of vertices into three parts consisting of a crown, a head, and a body, with the head separating the crown from the body. Under certain conditions concerning the crown and head vertices, which we will clarify in a moment, it is possible to show that there exists an optimal W-separator which contains the head vertices and reduces the given instance by removing the crown vertices. Recall that the parameter k in the decision variant of the W-separator asks for a W-separator of size at most k. Kumar and Lokshtanov [12] provide such a reducible structure and state that it is in a graph as long as the size of it is greater than $2kW$. The structure is called a (strictly) reducible pair and consists of crown and head vertices.

For an instance $G = (V, E)$ of the W-separator problem we say that $Y = \{y_v \in \mathbb{R}_{\geq 0}\}_{v \in V}$ is a *fractional W-separator* of G if Y is a feasible solution according to the LP formulation of the W-separator problem. It is not difficult to see that the objective of any optimal fractional W-separator is smaller than OPT, i.e., $\mathrm{LP}(G) \leq \mathrm{OPT}$. In principle, the LP objective is useful for finding a strictly reducible pair, since the head vertices in an optimal fractional W separator must have value one. Unfortunately, it is unknown whether each vertex that has value one in an optimal fractional W-separator is part of an optimal solution. This leads to the challenge of filtering out the right vertices, where the uncovered-objective - and in particular the structural properties of a strictly reducible pair - come into play.

Reducible Structure of the W-Separator Problem. In the following, we briefly summarize the definitions and theorems of [3, 7, 12]. For a vertex set $B \subseteq V$, denote by $\mathcal{B}$ the partitioning of B according to the connected components of $G[B]$.

Definition 4.2 ((strictly) reducible pair). For a graph $G = (V, E)$, a pair (A, B) of vertex disjoint subsets of V is a *reducible pair* if the following conditions are satisfied:

- $N(B) \subseteq A$.
- The size of each $C \in \mathcal{B}$ is at most W.
- There is an assignment function $g \colon \mathcal{B} \times A \to \mathbb{N}_0$, such that
  - for all $C \in \mathcal{B}$ and $a \in A$, if $g(C, a) \neq 0$, then $a \in N(C)$
  - for all $a \in A$ we have $\sum_{C \in \mathcal{B}} g(C, a) \geq 2W - 1$,
  - for all $C \in \mathcal{B}$ we have $\sum_{a \in A} g(C, a) \leq |C|$.

In addition, if there exists an $a \in A$ such that $\sum_{C \in \mathcal{B}} g(C, a) \geq 2W$, then (A, B) is a *strictly reducible pair*.

Next, we explain roughly the idea behind a reducible pair (A, B). The head and crown vertices correspond to A and B, respectively. That is, we want A to be part of our W-separator, and if that is the case, then no additional vertex from B is required to be in the solution since the components $C \in \mathcal{B}$ are isolated after removing A from G with $|C| \leq W$. Let $G = (V, E)$ be a graph. We say that $P_1, \ldots, P_m \subseteq V$ is a *$(W+1)$-packing* if for all $i, j \in [m]$ with $i \neq j$ the induced subgraph $G[P_i]$ is connected, $|P_i| \geq W+1$, and $P_i \cap P_j = \emptyset$. Note that for a W-separator $S \subseteq V$, it holds that $S \cap P_i \neq \emptyset$ for all $i \in [m]$. Thus, the size of a $(W+1)$-packing is a lower bound on the number of vertices needed for a W-separator.

Lemma 4.3 ([12], Lemma 17). *Let (A, B) be a reducible pair in G. There is a $(W+1)$-packing $P_1, \ldots, P_{|A|}$ in $G[A \cup B]$, such that $|P_i \cap A| = 1$ for all $i \in [|A|]$.*

Essentially, Lemma 4.3 provides a lower bound of $|A|$ vertices for a W-separator in $G[A \cup B]$. On the other hand, A is a W-separator of $G[A \cup B]$ while A separates B from the rest of the graph. This properties basically admits the following theorem.

Theorem 4.4 ([12], Lemma 18). *Let (G, k) be an instance of the W-separator problem, and (A, B) be a reducible pair in G. (G, k) is a yes-instance if and only if $(G - (A \cup B), k - |A|)$ is a yes-instance.*

Finally, we clarify why a strictly reducible pair exists if the size of G is larger than $2kW$. To do so, we make use of a lemma derivable from [3, 12]. A proof is given in the full version [1].

Lemma 4.5. *Let $G = (A \cup B, E)$ be a graph and $W \in \mathbb{N}_0$. Let $\mathcal{B}$ be the connected components given as vertex sets of $G[B]$, where for each $C \in \mathcal{B}$ we have $|C| \leq W$ and no $C \in \mathcal{B}$ is isolated, i.e., $N(C) \neq \emptyset$. If $|B| \geq (2W-1)|A|+1$, then there exists a non-empty strictly reducible pair (A', B'), where $A' \subseteq A$ and $B' \subseteq B$.*

We conclude the preliminary section with a lemma that connects strictly reducible pairs with the size of the graph.

Lemma 4.6 ([7], Lemma 6.14). *Let (G, k) be an instance of the W-separator problem, such that each component in G has size at least $W+1$. If $|V| > 2Wk$ and (G, k) is a yes-instance, then there exists a strictly reducible pair (A, B) in G.*

Running time analysis. Let (A, B) be a strictly reducible pair. We say (A, B) is a minimal strictly reducible pair if there does not exist a strictly reducible pair (A', B') with $A' \subset A$ and $B' \subseteq B$. Clearly, it can happen that reducible pairs arises after a reduction is executed. Therefore, we say $(A_1, B_1), \ldots, (A_m, B_m)$ is a *sequence of minimal strictly reducible pairs* if for all $i \in [m]$ the tuple (A_i, B_i) is a minimal strictly reducible pair in $G - \bigcup_{j=1}^{i-1} A_j$. Note that the definition of such a sequence implies that those tuples are pairwise disjoint, i.e., $(A_i \cup B_i) \cap (A_j \cup B_j) = \emptyset$ for all $i, j \in [m]$ with $i \neq j$. The proof of Theorem 4.1 can essentially be divided into three phases:

(1) Let $(A_1, B_1), \ldots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs in G, such that $G - \bigcup_{i \in [m]} A_i$ contains no minimal strictly reducible pair. The first phase is to show that after a polynomial number of iterations of `Global Semo Alt` with fitness f_2, a search point $X \in \{0,1\}^n$ exists in the population $\mathcal{P}$, such that $\mathrm{LP}(G) = |X_1| + \mathrm{LP}(G[u(X)])$ and there is a fractional optimal W-separator $Y = \{y_v \in \mathbb{R}_{\geq 0}\}_{v \in u(X)}$ with $y_v < 1$ for each $v \in V$. We will prove that in this case $G[u(X)]$ contains no strictly reducible pair, and that because of the equality relation $\mathrm{LP}(G) = |X_1| + \mathrm{LP}(G[u(X)])$ all the head vertices A_i for $i \in [m]$ are in X_1. That is, there is an optimal W-separator which contains a subset of X_1.
(2) The second phase is to filter $\bigcup_{i=1} A_i$ from $|X_1|$ so that we obtain a search point X' that selects only those as 1-bits. Once an X as described in Phase 1 is guaranteed to be in the population, the algorithm `Global Semo Alt` takes in expectation FPT-time to reach X'. Finally, it is important that X' remains in the population once we have found it. We show this by taking advantage of the structural properties of a reducible pair in combination with the uncovered-objective.
(3) For the last phase, we know by Lemma 4.6 already that $u(X')$ has size at most $2 \cdot \mathrm{OPT} \cdot W$. Once we ensure that X' is in $\mathcal{P}$ and stays there, we prove that `Global Semo Alt` finds in expectation an optimal solution in FPT-time.

In phase 1, we essentially make use of the LP objective. To prove that it works successfully, we will show the following two lemmas.

Lemma 4.7. *Using the fitness function f_2, the expected number of iterations of* `Global Semo Alt` *where the population $\mathcal{P}$ contains a search point $X \in \{0,1\}^n$ such that $LP(G) = LP(G[u(X)]) + |X_1|$ and there is an optimal fractional W-separator $\{y_v \in \mathbb{R}_{\geq 0}\}_{v \in u(X)}$ of $G[u(X)]$ with $y_v < 1$ for every $v \in u(X)$ is upper bounded by $O(n^3(\log n + OPT))$. Moreover, once $\mathcal{P}$ contains such a search point at any iteration, the same holds for all future iterations.*

Lemma 4.8. *Let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs in G, such that $G - \bigcup_{i=1} A_i$ contains no minimal strictly reducible pair. Let $X \in \{0,1\}^n$ be a sample, such that there is a an optimal fractional W-separator $\{y_v \in \mathbb{R}_{\geq 0}\}_{v \in u(X)}$ of $G[u(X)]$ with $y_v < 1$ for each $v \in u(X)$. If $|X_1| + LP(G[u(X)]) = LP(G)$, then $A_i \subseteq X_1$ and $B_i \cap X_1 = \emptyset$ for all $i \in [m]$.*

We guide the rest of this section by using $X \in \{0,1\}^n$ to denote a search point and $(A_1, B_1), \dots, (A_m, B_m)$ as a sequence of minimal strictly reducible pairs, where $A_{\texttt{all}} := \bigcup_{i=1}^m A_i$ and $B_{\texttt{all}} := \bigcup_{i=1}^m B_i$. For the Algorithm `Global Semo Alt` it is unlikely to jump from a uniformly random search point immediately to a search point satisfying Lemma 4.8. To guarantee a stepwise progress, we want that under the condition $\mathrm{LP}(G) = \mathrm{LP}(G[u(X)]) + |X_1|$ at each time $A_{\texttt{all}} \subsetneq X_1$ there exists a vertex of $v \in A_{\texttt{all}} \setminus X_1$ in an optimal fractional W-separator of $G[u(X)]$ which must have value one. For this purpose, the characterization of minimal strictly reducible pairs by optimal fractional W-separators is useful.

Lemma 4.9 ([7], Corollary 6.19 and Lemma 6.20). *Let $G = (V, E)$ be an instance of the W-separator problem and let $\{y_v \in \mathbb{R}_{\geq 0}\}_{v \in V}$ be an optimal fractional W-separator of G. If G contains a minimal strictly reducible pair (A, B), then $y_v = 1$ for all $v \in A$ and $y_u = 0$ for all $u \in B$.*

From Lemma 4.9 we can derive that if $(A_{\texttt{all}} \cup B_{\texttt{all}}) \cap X_1 = \emptyset$, such a vertex v must exist, but the question is what happens if the intersection is not empty. In particular, we want to avoid vertices of $B_{\texttt{all}}$ being in X_1, since reducible pairs in G may then no longer exist in $G[u(X)]$. We start with the proof of Lemma 4.7 and show later how it is related to a sequence of minimal strictly reducible pairs. The first lemma is a simple but useful observation.

Lemma 4.10. *For every $X \in \{0,1\}^n$ it holds that $LP(G) \leq |X_1| + LP(G[u(X)])$.*

If Lemma 4.10 is true, it is not difficult to derive that if we have that it holds with equality for a search point X, then $f_2(X)$ is a Pareto-optimal vector of the fitness function f_2, as given below as a corollary.

Corollary 4.11. *If a search point $X \in \{0,1\}^n$ satisfy $|X_1| + LP(G[u(X)]) = LP(G)$, then the vector $(|X_1|, *, LP(G[u(X)])$ is a Pareto-optimal vector of the fitness function f_2.*

The next lemma ensures that removing vertices with value one in an optimal fractional W-separator does not affect the objective of a fractional W-separator of the remaining graph.

Lemma 4.12 ([7], Corollary 6.17). *Let $G = (V, E)$ be an instance of the W-separator problem and let $\{y_v \in \mathbb{R}_{\geq 0}\}_{v \in V}$ be an optimal fractional W-separator of G. Let $V' \subseteq V(G)$, such that $y_v = 1$ for all $v \in V'$. Then, $\{y_v \mid v \in V \setminus V'\}$ is an optimal fractional W-separator of $G - V'$, i.e., $\sum_{v \in V \setminus V'} y_v = LP(G - V')$.*

Corollary 4.11 and Lemma 4.12 allow incremental progress in the set of 1-bits with respect to search points $X \in \mathcal{P}$ that satisfy $|X_1| + \mathrm{LP}(G[u(X)]) = \mathrm{LP}(G)$ without backstepping. With this ingredient we can prove Lemma 4.7 (see full version [1] for a proof). Since f_3 has one less objective than f_2, one can derive the following lemma.

Lemma 4.13. *Using the fitness function f_3, the expected number of iterations of* `Global Semo Alt` *where the population $\mathcal{P}$ contains no search point $X \in \{0,1\}^n$ such that $LP(G) = LP(G[u(X)]) + |X_1|$ and there is an optimal fractional W-separator $\{y_v \in \mathbb{R}_{\geq 0}\}_{v \in u(X)}$ of $G[u(X)]$ with $y_v < 1$ for every $v \in u(X)$ is upper bounded by $O(n^2(\log n + OPT))$.*

Our next goal is to prove Lemma 4.8. To identify the head vertices $A_{\texttt{all}}$ with respect to an optimal fractional W-separator, we want to ensure that the selection of the vertices of $B_{\texttt{all}}$ are distinguishable so that it cannot come to a conflict with Lemma 4.7. To do this, we will make use of the LP-objective and show that for a search point X with $X_1 \cap B_{\texttt{all}} \neq \emptyset$ we have $\mathrm{LP}(G) < \mathrm{LP}(G[u(X)]) + |X_1|$. Let (A, B) be a minimal strictly reducible pair in G. The essential idea is to use $(W+1)$-packings in $G[A \cup B]$, since they provide lower bounds for W-separators. From Lemma 4.3 one can deduce that $G[A \cup B]$ contains a maximum $(W+1)$-packing $\mathcal{Q}$ of size $|A|$, since every vertex of A is contained exactly in one element of $\mathcal{Q}$. Inspired by ideas on how to find crown decompositions in weighted bipartite graphs from [3, 12], we prove that removing vertices from B only partially affects the size of the $(W+1)$-packing in $G[A \cup B]$, as stated in the following lemma.

Lemma 4.14. *Let (A, B) be a minimal strictly reducible pair in $G = (V, E)$ and let $S \subset A \cup B$ with $|S| \leq |A|$. If $S \cap B \neq \emptyset$, then $G[(A \cup B)] - S$ contains a packing of size $|A| - |S| + 1$.*

In contrast, note that removing vertices $S \subseteq A$ from $G[A \cup B]$ would decrease the size of a $(W+1)$-packing by $|S|$, i.e., a maximum $(W+1)$-packing in $G[A \cup B] - S$ has size $|A| - |S|$. We moved the proof of Lemma 4.14 to the full version [1], since it is more technical and too long given the space constraints. Essentially, we make use of the following two lemmas and properties of network flows. In particular, these lemmas describe the new properties we have found for minimal strictly reducible pairs and may be of independent interest.

Lemma 4.15. *Let (A, B) be a minimal strictly reducible pair in G with parameter W. Then, for every $a^* \in A$ there is an assignment function $g\colon \mathcal{B} \times A \to \mathbb{N}_0$ like in Definition 4.2 that satisfies $\sum_{C \in \mathcal{B}} g(C, a^*) \geq 2W$ and $\sum_{C \in \mathcal{B}} g(C, a) \geq 2W - 1$ for every $a \in A \setminus \{a^*\}$.*

Concerning Lemma 4.15, we remark that the new feature to before is that the particular vertex (in the lemma a^*) can be chosen arbitrarily.

Lemma 4.16. *Let (A, B) be a minimal strictly reducible pair in G with parameter W. Then, for every $A' \subseteq A$ we have $|V(\mathcal{B}_{A'})| \geq |A'|(2W - 1) + 1$.*

To conclude the Phase 1 we need to prove Lemma 4.8. Equipped with Lemma 4.14 we may prove statements about the LP-objective if $X_1 \cap B_{\texttt{all}} \neq \emptyset$. In doing so, we prove another relation with respect to such a sequence, which fits the proof and will be useful later.

Lemma 4.17. *Let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs and let $X \in \{0,1\}^n$ be a sample.*

(1) If $X_1 = \bigcup_{i=1}^m A_i$, then $LP(G) = LP(G[u(X)]) + |X_1|$.
(2) If $X_1 \cap B_\ell \neq \emptyset$ for an $\ell \in [m]$, then $LP(G(u[X])) + |X_1| > LP(G)$.

Suitable for Lemma 4.7 we have characterized the case $X_1 \cap B_{\mathtt{all}} \neq \emptyset$. It remains to give a relation to this lemma when $A_{\mathtt{all}} \cap X_1 \neq \emptyset$ and $A_{\mathtt{all}} \subsetneq X_1$. In particular, we want to ensure that in this case at least one vertex of $A_{\mathtt{all}} \setminus X_1$ must be one in an optimal fractional W-separator of $G[u(X)]$.

Lemma 4.18. *Let (A, B) be a minimal strictly reducible pair in G and let $\hat{A} \subset A$. Then, there is a partition $A_1, \dots, A_m$ of $A \setminus \hat{A}$ with disjoint vertex sets $B_1, \dots, B_m \subseteq B$, such that for each $i \in [k]$ the tuple (A_i, B_i) is a minimal strictly reducible pair in $G - \hat{A}$.*

By Lemma 4.9 we already know that the head vertices of a minimal strictly reducible pair in an optimal fractional W-separator have value one. Lemma 4.18 ensures that if some of the head vertices are removed, the value of the remaining head vertices in the respective optimal fractional solution remain one. The proof of Lemma 4.8 can be found in the the full version [1] and concludes Phase 1.

Next, we prove that Phase 2 works successfully. After Phase 1, we have a search point X in the population $\mathcal{P}$ with $A_{\mathtt{all}} \subseteq X_1$ such that $\mathrm{LP}(G) = \mathrm{LP}(G[u(X)]) + |X_1|$. Consequently, $|X_1| \leq \mathrm{OPT}$ and therefore we can prove that `Global Semo Alt` reaches a search point X' with $X'_1 = A_{\mathtt{all}}$ from X in FPT-time.

Lemma 4.19. *Let $G = (V, E)$ be an instance of the W-separator problem, and let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs in G, such that $G - \bigcup_{i=1} A_i$ contains no strictly reducible pair. Using the fitness function f_2, the expected number of iterations of* `Global Semo Alt` *until the population $\mathcal{P}$ contains a search point X with $X_1 = \bigcup_{i=1}^m A_i$ is upper bounded by $O\left(n^3(\log n + OPT) + n^2 \cdot 2^{OPT}\right)$.*

The question that remains is whether we keep X' in the population once we find it. This is where the uncovered-objective and the structural properties of minimal strictly reducible pairs come into play.

Lemma 4.20. *Let $X \in \{0,1\}^n$ and let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs in G, such that $G - \bigcup_{i=1} A_i$ contains no strictly reducible pair. If $X_1 = \bigcup_{i=1}^m A_i$, then X is a Pareto-optimal solution.*

We are ready for the final theorem of this section, which shows that Phase 3 also works successfully.

Proof of Theorem 4.1: Let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs, such that $G - \bigcup_{i=1}^m A_i$ contains no strictly reducible pair. Furthermore, let $\mathcal{P}$ be a population with respect to f_2 in the algorithm `Global Semo Alt`. By Lemma 4.19 we have a search point $X \in \mathcal{P}$ with $X_1 = \bigcup_{i=1}^m A_i$ after $O\left(n^3(\log n + \mathrm{OPT}) + n^2 \cdot 2^{\mathrm{OPT}}\right)$ iterations in expectation. Moreover, by Lemma 4.20 X is a Pareto-optimal solution, and by Theorem 4.4 there is an optimal W-separator V^* such that $X_1 \subseteq V^*$.

Since $G[u(X)]$ contains no strictly reducible pair, we can derive from Lemma 4.6 that $|V(G[u(X)])| \leq 2 \cdot \mathrm{OPT} \cdot W$. The algorithm `Global Semo Alt` calls with $1/3$ probability the mutation that flips every vertex $u(X)$ with $1/2$ probability in X. That is, reaching a state X' from X, such that $X'_1 = V^*$ has a probability of at least $\Omega\left(2^{-2 \cdot \mathrm{OPT} \cdot W}\right)$, where selecting X' in $\mathcal{P}$ has probability $\Omega(1/n^2)$ (cf. Lemma 2.1). Thus, once X is contained in $\mathcal{P}$ it takes in expectation $O\left(n^2 \cdot 4^{\mathrm{OPT} \cdot W}\right)$ iterations reaching X'. As a result, the algorithm needs in total $O\left(n^3(\log n + \mathrm{OPT}) + n^2 \cdot 4^{\mathrm{OPT} \cdot W}\right)$ iterations finding an optimal W-separator in expectation.

5 APPROXIMATIONS

In this section we consider the W-separator problem with the fitness f_2 and f_3 associated with `Global Semo` and `Global Semo Alt`. We show that the algorithms find approximate solutions when we reduce their overhead. In particular, we prove the following theorems.

Theorem 5.1. *Using the the fitness function f_3, the expected number of iterations of* `Global Semo` *until it finds a $(W+1)$-approximation in $G = (V, E)$ is upper bounded by $O\left(n^2(\log n + W \cdot OPT)\right)$.*

Theorem 5.2. *Let $G = (V, E)$ be an instance of the W-separator problem and let $\varepsilon \in [0, 1)$.*

(1) Using the fitness function f_2, the expected number of iterations of `Global Semo Alt` *until an $(1 + \varepsilon(3/2W - 1/2))$-approximation is sampled is upper bounded by $O\left(n^3(\log n + W \cdot OPT) + 2^{OPT} + n^2 \cdot 4^{(1-\varepsilon)OPT \cdot W}\right)$.*
(2) Using the fitness function f_3, the expected number of iterations of `Global Semo Alt` *until a $(2 + \varepsilon(3/2W - 1/2))$-approximation is sampled is upper bounded by $O\left(n^2(\log n + W \cdot OPT) + n \cdot 4^{(1-\varepsilon)OPT \cdot W}\right)$.*

Note that Theorem 5.2 implies that we can hope for incremental progress towards an optimal solution if we compare it to Theorem 4.1. Note also that Theorem 5.2 (1) has a running time of $O\left(n^3(\log n + W \cdot \mathrm{OPT}) + n^2 \cdot 4^{(1-\varepsilon)\mathrm{OPT} \cdot W}\right)$ if $\varepsilon < 1/2$.

To prove our theorems, we show that once there is a search point in the population that has a desired target value with respect to the LP-objective and the one-objective, then the algorithms find in polynomial time a W-separator that does not exceed this target value. That is, the 1-bits of this search point do not necessarily have to form a W-separator.

Lemma 5.3. *Let $G = (V, E)$ be an instance of the W-separator problem, $\mathcal{P}$ a population with respect to the fitness function f_2 or f_3, $c > OPT$, and $X \in \mathcal{P}$ a search point satisfying $|X_1| + (W + 1) \cdot LP(G[u(X)]) \leq c$. Using the the fitness function f_2 or f_3, the expected number of iterations of* `Global Semo` *until it finds a W-separator S in G with $|S| \leq c$ is upper bounded by $O\left(n^2 W \cdot OPT\right)$ or $O\left(n^3 W \cdot OPT\right)$, respectively.*

We conclude this section with the proof of Theorem 5.2 (1). Thereby, we basically need to show that we reach in the stated runtime a search point X that satisfies the precondition of Lemma 5.3 with the desired approximation value.

Proof of Theorem 5.2 (1). Let $X \in \{0,1\}^n$ be a search point such that $G[u(X)]$ contains no minimal strictly reducible pair (*irreducible-condition*). Furthermore, let S be an optimal solution of $G[u(X)]$ and let $U = u(X) \setminus S$. Note that $|S| \leq \mathrm{OPT}$. Since $G[u(X)]$ contains no minimal strictly reducible pair, we have $|U| = |u(X)| - |S| \leq 2W|S| - |S| = |S|(2W-1)$ by Lemma 4.6.

Recall that `Global Semo Alt` chooses with $1/3$ probability the mutation that flips every bit corresponding to the vertices in $u(X)$ with $1/2$ probability. From this, the search point X has a probability of $\Omega\left(2^{-(1-\varepsilon)|S|-(1-\varepsilon)|S|\cdot(2W-1)}\right) = \Omega\left(4^{-(1-\varepsilon)|S|\cdot W}\right)$ to flip $(1-\varepsilon)|S|$ fixed vertices of S and to not flip $(1-\varepsilon)|S|\cdot(2W-1)$ fixed vertices of U in one iteration. Independently from this, half of the remaining vertices of S and U are additionally flipped in this iteration, i.e., $\frac{1}{2}\varepsilon|S|$ of S and $\frac{1}{2}\varepsilon|U|$ of U. Let S' and U' be the flipped vertices in this iteration and let X' be the according search point. Note that $\mathrm{LP}(X') \leq |S| - |S'|$ simply because there is a W-separator of $G[u(X)] - S'$ of size $|S| - |S'|$. Hence, we have

$$
\begin{aligned}
&|X_1'| + (W+1)\cdot \mathrm{LP}(G[u(X')]) \\
&= |X_1| + |S'| + |U'| + (W+1)\cdot \mathrm{LP}(G[u(X')]) \\
&\leq |X_1| + |S'| + |U'| + (W+1)\cdot (|S| - |S'|) \\
&= |X_1| + |S|(W+1) - |S'|W + |U'|.
\end{aligned}
$$

Next, we upper bound $|S'|$ and $|U'|$ in terms of $|S| \leq \mathrm{OPT}$. Using the fact $|U| \leq |S|(2W-1)$, we obtain $|U'| = \frac{1}{2}\varepsilon|U| \leq \frac{1}{2}\varepsilon|S|(2W-1) = \varepsilon|S|W - \frac{1}{2}\varepsilon|S|$. Regarding S' we have $|S'| = (1-\varepsilon)|S| + \frac{1}{2}\varepsilon|S| = |S| - \varepsilon|S| + \frac{1}{2}\varepsilon|S|$. As a result, we obtain

$$
\begin{aligned}
&|X_1'| + (W+1)\cdot \mathrm{LP}(G[u(X')]) \\
&\leq |X_1| + |S|(W+1) - |S'|W + |U'| \\
&\leq |X_1| + |S|(W+1) - (|S| - \varepsilon|S| + \frac{1}{2}\varepsilon|S|)W + \varepsilon|S|W - \frac{1}{2}\varepsilon|S| \\
&\leq |X_1| + |S|(W+1) - |S|W + \varepsilon|S|W - \frac{1}{2}\varepsilon|S|W + \varepsilon|S|W - \frac{1}{2}\varepsilon|S| \\
&= |X_1| + |S| + |S|\left(2\varepsilon W - \frac{1}{2}\varepsilon W - \frac{1}{2}\varepsilon\right).
\end{aligned}
$$

Observe that once a desired X is guaranteed to be in the population, an event described above occurs after $O\left(n^2 \cdot 4^{(1-\varepsilon)|S|\cdot W}\right)$ iterations in expectation for the fitness functions f_2, where the factor n^2 comes from selecting X (cf. Lemma 2.1).

Let $(A_1, B_1), \ldots, (A_m, B_m)$ be a sequence of minimal strictly reducible pairs, such that $G - \bigcup_{i=1}^m A_i$ contains no strictly reducible pair. By Lemma 4.19 we have a search point X in the population $\mathcal{P}$ with $X_1 = \bigcup_{i=1}^m A_i$ after $O\left(n^3(\log n + \mathrm{OPT}) + n^2 \cdot 2^{\mathrm{OPT}}\right)$ iterations in expectation. Note that X satisfies the irreducible-condition. Moreover, X is a Pareto-optimal solution by Lemma 4.20. By Theorem 4.4 we have $|X_1| = \mathrm{OPT} - |S|$. Using that $|S| \leq \mathrm{OPT}$, we obtain

$$
\begin{aligned}
&|X_1'| + (W+1)\cdot \mathrm{LP}(G[u(X')]) \\
&\leq |X_1| + |S| + |S|\left(2\varepsilon W - \frac{1}{2}\varepsilon W - \frac{1}{2}\varepsilon\right) \\
&= \mathrm{OPT} - |S| + |S| + |S|\left(2\varepsilon W - \frac{1}{2}\varepsilon W - \frac{1}{2}\varepsilon\right) \\
&\leq \mathrm{OPT}\left(1 + \varepsilon\left(\frac{3}{2}W - \frac{1}{2}\right)\right).
\end{aligned}
$$

As a result, by the choice of X the resulting search point X' satisfies the precondition of Lemma 5.3 with $c = \mathrm{OPT}\left(1 + \varepsilon\left(\frac{3}{2}W - \frac{1}{2}\right)\right)$. That is, once X' is in the population $\mathcal{P}$, the algorithm `Global Semo Alt` need in expectation $O(n^3 W \cdot \mathrm{OPT})$ iterations having a search point in $\mathcal{P}$ which is a $\left(1 + \varepsilon\left(\frac{3}{2}W - \frac{1}{2}\right)\right)$-approximation. In summary, in expectation the desired search point X' is in $\mathcal{P}$ after $O\left(n^3(\log n + W \cdot \mathrm{OPT}) + n^2 \cdot 2^{\mathrm{OPT}} + n^2 \cdot 4^{(1-\varepsilon)\mathrm{OPT}\cdot W}\right)$ iterations. □

6 CONCLUSION

In this work, we studied the behavior of evolutionary algorithms with different multi-objective fitness functions for the W-separator problem from the perspective of parameterized complexity. More precisely, we investigated the running time of such evolutionary algorithms depending on the problem parameter $\mathrm{OPT} + W$. Our analysis was based on properties of reducible structures, showing that, given a suitable fitness function, the evolutionary algorithm tends to reduce the given instance along these structures. Once this is done, the running time for either obtaining an arbitrarily close approximation or an exact solution is tractable with respect to the problem parameter. In particular, this shows that evolutionary algorithms solve the W-separator problem in expectation in FPT-time for the parameter $\mathrm{OPT} + W$.

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