

Adjacent vertex distinguishing total coloring in split graphs

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ABSTRACT

An adjacent vertex distinguishing (AVD-) total coloring of a graph G is a total coloring such that any two adjacent vertices u and v have distinct sets of colors, that is, $C(u) \neq C(v)$, where $C(v)$ is the set of colors of the edges incident to v and the color of v . The adjacent vertex distinguishing (AVD)-total chromatic number of a graph G , $\chi_a''(G)$ is the minimum integer k such that there exists an AVD-total coloring of G using k colors. It is known that $\chi_a''(G) \geq \Delta + 1$, where Δ is the maximum degree of the graph. The AVD-total coloring conjecture states that for any graph G , $\chi_a''(G) \leq \Delta + 3$. In this paper, we study AVD-total coloring in split graphs. We verify the AVD-total coloring conjecture for split graphs and classify certain classes of split graphs according to their AVD-total chromatic number.

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1. Introduction

All the graphs considered in this paper are finite, simple, and undirected. The graph obtained by deleting the vertices of a set $C = \{u_1, u_2, \dots, u_k\}$ together with all the edges incident with these vertices is denoted by $G - C$. Similarly, the graph obtained by deleting the edges of a set $D = \{e_1, e_2, \dots, e_t\}$ is denoted by $G - D$. The degree of a vertex v in G is the number of edges incident to the vertex v , and it is denoted by $d_G(v)$. The maximum degree of a graph G is denoted by $\Delta(G)$ and $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$. Throughout the paper, Δ denotes the maximum degree of graph G . A set of vertices is called *independent* if no two vertices in the set are adjacent, and a set of vertices is called a *clique* if every two vertices in the set are adjacent. The *clique number* of a graph G , denoted by $\omega(G)$, is the maximum integer k such that a clique of size k is contained as a subgraph of G . A graph G is called a *split graph*, denoted by (S, R) if its vertex set can be partitioned into two sets S and R such that S induces a clique, and R induces an independent set. A *complete split graph* is a split graph such that each vertex of R is adjacent to every vertex in the clique S .

A proper *total k -coloring* of a graph G is a mapping $\phi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that (i) for every two adjacent vertices u and v , $\phi(u) \neq \phi(v)$, (ii) for any edge e incident to a vertex u , $\phi(e) \neq \phi(u)$, and (iii) for any two edges e_1 and e_2 incident to a common vertex, $\phi(e_1) \neq \phi(e_2)$. Throughout the paper, whenever we say total coloring, we mean a proper total coloring. The *total chromatic number* of a graph G is the minimum integer k such that there exists a total k -coloring of the graph G and it is denoted by $\chi''(G)$. Observe that for any graph G , $\chi''(G) \geq \Delta + 1$, where Δ is the maximum degree of G . Behzad [1] and Vizing [15] independently posed the *total coloring conjecture* which states that for any graph G , $\chi''(G) \leq \Delta + 2$.

An *adjacent vertex distinguishing (AVD-) total coloring* of a graph G is a total coloring, say ϕ , with an additional property, called AVD-property, that is, for any pair of adjacent vertices u and v , $C_\phi(u) \neq C_\phi(v)$, where $C_\phi(x)$ denotes the color set

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of used colors on vertex x with respect to the coloring ϕ and $C_\phi(x) = \{\phi(x)\} \cup \{\phi(xy) : y \in N(x)\}$, for $x \in V(G)$. Throughout the paper, we drop the suffix and denote the color set of any vertex v with respect to an AVD-total coloring as $C(v)$ unless there is more than one coloring used. We denote the set of available colors for a vertex v as $\bar{C}(v)$, that is, the set of colors that have not been used on the vertex v and the edges incident to it. Observe that, if the AVD-property holds for a pair of adjacent vertices u and v , then $\bar{C}(u) \neq \bar{C}(v)$. The minimum integer k , such that G has an AVD-total coloring using k colors, is the *AVD-total chromatic number* of G denoted by $\chi''_a(G)$. This concept was introduced by Zhang et al. [20]. Clearly, for a graph G , $\chi''_a(G) \geq \Delta + 1$. Observe that given a pair of adjacent vertices u and v such that $d(u) = d(v) = \Delta$, at least $\Delta + 2$ colors are required to totally color the neighborhoods of u and v so that $C(u) \neq C(v)$. Therefore, we have the following result:

Theorem 1.1. [20] *If a graph G has two vertices of maximum degree which are adjacent, then $\chi''_a(G) \geq \Delta + 2$.*

Zhang et al. [20] also posed the *AVD-total coloring conjecture* similar to the total coloring conjecture.

AVD-total coloring conjecture: For a graph G with maximum degree Δ ,

$$\chi''_a(G) \leq \Delta + 3.$$

The conjecture has been verified for general graphs with $\Delta = 3$ by Wang [16] and Chen [5]. AVD-total coloring has been further studied for graphs with $\Delta = 3$ in [9,12]. Papaioannou and Raftopoulou [13] proved the AVD-total coloring conjecture for 4-regular graphs. Recently, Lu et al. [11] validated the conjecture for all graphs with $\Delta = 4$. The AVD-total coloring conjecture is also known to be true for many families of graphs such as complete graphs [20], hypercubes [4], indifference graphs [14], planar graphs with $\Delta \geq 8$ [2,6-8,17,18] and outerplanar graphs [19].

Theorem 1.2. [20] *For a complete graph K_n with n vertices,*

$$\chi''_a(K_n) = \begin{cases} n + 1, & \text{if } n \text{ is even,} \\ n + 2, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.3. [10] *If G is a bipartite graph, then G is a Δ -edge colorable graph, where Δ is the maximum degree of G .*

In [3], Chen et al. verified the total coloring conjecture for split graphs and proved that for a split graph G , $\chi''(G) = \Delta + 1$ if Δ is even. In this paper, we study AVD-total coloring in split graphs. We deploy the same technique used in the paper [3] to prove the AVD-total coloring conjecture for split graphs. We also classify the split graphs $G = (S, R)$ such that in the bipartite graph $B = G - E(G[S])$, the degree of any vertex in R is less than the maximum degree of any vertex in S with respect to their AVD-total chromatic number. In particular, we prove that for a split graph $G = (S, R)$ with $|S| = n$ and $\Delta \geq \max\{d(u) : u \in R\} + n$, if there exist two vertices with degree Δ in S then $\chi''_a(G) = \Delta + 2$; otherwise $\chi''_a(G) = \Delta + 1$. Furthermore, we classify the complete split graphs according to their AVD-total chromatic number.

2. Verification of AVD-total coloring conjecture for split graphs

In this section, we verify the AVD-total coloring conjecture for split graphs. First, we need some definitions.

A *Latin rectangle* of size $r \times k$ (where $r \leq k$), is an $r \times k$ array based on the elements $1, 2, \dots, k$ such that each element occurs exactly once in each row and at most once in each column. A *Latin square* of order k is a $k \times k$ array based on the elements $1, 2, \dots, k$ such that each element occurs exactly once in each row and exactly once in each column. A Latin square $M = [m_{i,j}]$ is said to be *commutative* if $m_{i,j} = m_{j,i}$, for $1 \leq i, j \leq k$ and M is said to be *idempotent* if $m_{i,i} = i$, for $1 \leq i \leq k$. It is well known that an idempotent commutative Latin square (ICLS) of order k exists if and only if k is odd. If the rows of the Latin square are just cyclic permutations (one shift of the elements to the right) of the previous row, then the Latin square is said to be *circulant* (anti-circulant, if the cyclic permutations are actually left shifts). Let $M(k) = [m_{i,j}]$ where $m_{i,j} \equiv (i + j)k \pmod{2k - 1}$, $1 \leq m_{i,j} \leq 2k - 1$, for $1 \leq i, j \leq 2k - 1$. Observe that $M(k)$ (in Fig. 1) is an ICLS of order $2k - 1$ and the Latin square $M(k)$ is anti-circulant as well.

Definition 2.1. [3] A *color diagram*, $\mathcal{C} = (R_1, R_2, \dots, R_n)$ of frame $d = (d_1, d_2, \dots, d_n)$ is an ordered set of color arrays, where each color array $R_i = [c_{i,1}, c_{i,2}, \dots, c_{i,d_i}]$ is of length d_i and consists of different colors for $1 \leq i \leq n$. A color diagram is called *monotonic* if the color $c_{i,j}$ occurs at most $d_i - j$ times in R_1, R_2, \dots, R_{i-1} for all i , $1 \leq i \leq n$ and $1 \leq j \leq d_i$. Let $\mathcal{C}(n, k)$ denote the color diagram (R_1, R_2, \dots, R_n) , if the colors $c_{i,j}$ in each R_i ($1 \leq j \leq d_i$), are the entries of $M(k)$ and $n \leq 2k - 1$.

Definition 2.2. [3] Let $B = (U, V, E)$ be a bipartite graph. An *edge coloring scheme* with respect to U is a collection of sets of colors $\mathcal{S} = \{S_u\}_{u \in U}$, where S_u is a set of $d_B(u)$ distinct colors for $u \in U$. Scheme \mathcal{S} is *fulfilled* by an edge coloring of B if the set of colors occurring on edges incident to u is exactly S_u for all $u \in U$. An edge coloring scheme \mathcal{S} can be arranged into a color diagram by assigning an order to colors in S_u for all $u \in U$ and an order to $\{S_u\}_{u \in U}$.

1	$k + 1$	2	$k + 2$...	$2k - 1$	k
$k + 1$	2	$k + 2$	3	...	k	1
2	$k + 2$	3	$k + 3$...	1	$k + 1$
...
...
$2k - 1$	k	1	$k + 1$...	$2k - 2$	$k - 1$
k	1	$k + 1$	2	...	$k - 1$	$2k - 1$

Fig. 1. An idempotent commutative Latin square of order $(2k - 1)$.

Lemma 2.1. [3] *If S can be arranged into a monotonic color diagram, then B has an edge coloring that fulfills S .*

Next, we provide AVD-total colorings for split graphs.

Proposition 2.1. *Let $G = (S, R)$ be a split graph. If Δ is odd, then $\chi''_a(G) \leq \Delta + 2$.*

Proof. Let $S = \{u_1, u_2, \dots, u_n\}$ and $R = \{v_1, v_2, \dots, v_m\}$. Assume that $\Delta = n - 1 + \Delta_S$, where Δ_S is the maximum degree of any vertex in S in the bipartite graph $B = G - E(G[S])$. Observe that $d_G(u_i) \geq d_G(v_j)$ for all vertices $u_i \in S$ and $v_j \in R$. Since Δ is odd, there exists an ICLS of order $\Delta + 2$. We take an ICLS $M(k) = [m_{i,j}]$ of order $2k - 1$ such that $m_{i,j} \equiv (i + j)k \pmod{2k - 1}$, where $k = \frac{\Delta + 3}{2}$. First, we totally color the subgraph $G[S]$ from $M(k)$. Define an AVD-total coloring π of G such that $\pi(u_i) = m_{i,i}$, for $1 \leq i \leq n$ and $\pi(u_i v_j) = m_{i,j}$, for $1 \leq i, j \leq n, i \neq j$. Next, to complete the AVD-total coloring π , we have to properly color the edges of the bipartite graph $B = G - E(G[S])$. Note that each vertex u_i has $\Delta + 2 - n$, that is, $\Delta_S + 1$ available colors, and the set of available colors is $\{m_{i,j} : n + 1 \leq j \leq \Delta + 2\}$. We color the remaining edges incident to each vertex u_i with colors $S_{u_i} = \{m_{i,j} : n + 1 \leq j \leq \Delta + 1\}$. The color diagram $\mathcal{C}(n, k)$ is an arrangement of an edge coloring scheme $\mathcal{S} = \{S_{u_i}\}_{i=1}^n$. By the definition of $M(k)$, $m_{i,j} = m_{i-1, j+1}$, for $1 \leq i, j \leq 2k - 1$. It follows that the color diagram $\mathcal{C}(n, k) = (S_{u_1}, S_{u_2}, \dots, S_{u_n})$ is monotonic. Since $\mathcal{C}(n, k)$ is monotonic, by Lemma 2.1 there is an edge coloring π_1 of the bipartite graph B which fulfills S . Let $\pi(u_i v_j) = \pi_1(u_i v_j)$ for all $u_i v_j \in E(B)$. Now we only have to color the vertices of R . If n is odd, then $\pi(v_j) = m_{1, n-1}$ for each vertex $v_j \in R$ as the color $m_{1, n-1}$ is not present in the color diagram $\mathcal{C}(n, k)$ and so it is not present on any edge incident to v_j . If n is even, then $\pi(v_j) = m_{1, n}$ for each vertex $v_j \in R$ as the color $m_{1, n}$ is not present on any edge incident to v_j . Thus we get a proper total coloring. Note that the set of available colors on each vertex u_i , $\tilde{C}(u_i)$ contains the color $m_{i, \Delta + 2}$, which is different for each $i, 1 \leq i \leq n$. Since $M(k)$ is anti-circulant and each set $\tilde{C}(u_i)$ contains a different color $m_{i, \Delta + 2}$, $\tilde{C}(u_i) \neq \tilde{C}(u_j)$ for every pair of vertices u_i and u_j . Moreover, for any adjacent pair u_i and v_j , $\tilde{C}(u_i) \neq \tilde{C}(v_j)$ by the definition of $M(k)$ as $M(k)$ is a commutative anti-circulant Latin square. Hence, the obtained coloring π is an AVD-total coloring of G using $\Delta + 2$ colors. \square

Proposition 2.2. *Let $G = (S, R)$ be a split graph such that Δ is even. If there is only one vertex of degree Δ in graph G , then $\chi''_a(G) = \Delta + 1$; otherwise $\chi''_a(G) \leq \Delta + 3$.*

Proof. Let $|S| = n, |R| = m, S = \{u_1, u_2, \dots, u_n\}$, and $R = \{v_1, v_2, \dots, v_m\}$. Assume that $\Delta = n - 1 + \Delta_S$, where Δ_S is the maximum degree of any vertex in S in the bipartite graph $B = G - E(G[S])$. Observe that $d_G(u_i) \geq d_G(v_j)$ for $u_i \in S$ and $v_j \in R$. We consider the following two cases:

Case 1: There is a unique vertex with degree Δ .

Let $u_i \in S$ such that $d(u_i) = \Delta$. We take some edge $u_i v_j$ and remove it from G , say $G' = G - \{u_i v_j\}$. Therefore, $\Delta(G') = \Delta - 1$. Note that $\Delta(G')$ is odd. Therefore, by Proposition 2.1, there exists an AVD-total coloring of G' using $\Delta(G') + 2$ colors. We obtain an AVD-total coloring π of G' by using an ICLS M of order $\Delta(G') + 2$ as we obtain in the proof of Proposition 2.1. Finally, we color the edge $u_i v_j$. Note that the set of available colors on each vertex u_i , $\tilde{C}(u_i)$ contains the color $m_{i, \Delta(G') + 2}$, which is different for each $i, 1 \leq i \leq n$. Therefore, we assign $\pi(u_i v_j) = m_{i, \Delta(G') + 2}$. Thus, we obtain an AVD-total coloring of G using $\Delta(G') + 2$ colors, that is, $\Delta(G) + 1$ colors. Hence, $\chi''_a(G) = \Delta + 1$.

Case 2: There exist more than one vertex with degree Δ .

In this case, we give an AVD-total coloring of G using $\Delta + 3$ colors. Since Δ is even, there exists an ICLS of order $\Delta + 3$. We take an ICLS $M(k) = [m_{i,j}]$ of order $2k - 1$ such that $m_{i,j} \equiv (i + j)k \pmod{2k - 1}$, where $k = \frac{\Delta + 4}{2}$. Similar to Case 1, we

first totally color the graph induced on S from the Latin subsquare $[m_{i,j}]_{1 \leq i, j \leq n}$ and then we color the edges of the bipartite graph $B = G - E(G[S])$ from the remaining colors $\{m_{i,j} : n + 1 \leq i \leq \Delta + 2, 1 \leq j \leq n\}$. Finally, we color the vertices of R . Thus, in a similar way to Case 1, we obtain an AVD-total coloring using $\Delta + 3$ colors. Hence, $\chi''_a(G) \leq \Delta + 3$. \square

Next result immediately follows from Proposition 2.1 and Proposition 2.2:

Theorem 2.1. *The AVD-total coloring conjecture is true for split graphs.*

3. Classification of split graphs

In this section, we study the AVD-total chromatic number of split graphs.

Observation 3.1. *Let $G = (S, R)$ be a split graph such that G is not a complete graph and $|S| = n$. Then we have the following:*

1. $d(u) \geq n - 1$, for every vertex $u \in S$ and $d(v) \leq n$, for every vertex $v \in R$.
2. A maximum degree vertex of G always belongs to S only.

Proposition 3.1. *Let $G = (S, R)$ be a split graph such that $|S| = n$, n is even and $\Delta \geq \max\{d(v) : v \in R\} + n$. If there exists exactly one vertex of degree Δ , then $\chi''_a(G) = \Delta + 1$; otherwise, $\chi''_a(G) = \Delta + 2$.*

Proof. Let $\Delta = n - 1 + \Delta_S$, where Δ_S is the maximum degree of any vertex in S in the bipartite graph $B = G - E(G[S])$. Let $\Delta_S = k$. Since n is even, $\chi''_a(G[S]) = n + 1$. Let $C = \{1, 2, \dots, n + 1\}$ be a set of $n + 1$ colors and f be an AVD-total coloring of $G[S]$ which uses colors from the set C . Now we shall extend the coloring f to get an AVD-total coloring of the graph G . Next, we have to color the edges between the sets R and S such that this coloring, together with the AVD-coloring f , is an AVD-total coloring.

Case 1: There is only one vertex in S with degree Δ .

Let $v \in S$ such that $d(v) = \Delta$. Therefore, v has k neighbors in R . Note that the vertex v has one available color from the set of colors C , let c be the color available on v . We color one edge e incident to v from B with color c , that is, $f(e) = c$. Now consider the graph $G' = G - E(G[S]) - \{e\}$. Observe that the graph G' is a bipartite graph with maximum degree $k - 1$. We know that G' can be edge colored with $k - 1$ colors, by Theorem 1.3. Let $C' = \{n + 2, \dots, n + k\}$ be a set of $k - 1$ new colors. Let f' be the edge coloring of G' using the colors from the set C' . Let $f(u_i v_j) = f'(u_i v_j)$, for any edge $u_i v_j \in E(G)$ such that $u_i \in S$ and $v_j \in R$. Next, color the vertex $v_j \in R$ with any available color which has not been used for the edges incident to v_j and for the neighbors of v_j . Observe that the AVD-property still holds for any pair of vertices in S , as we started with an AVD-total coloring f and the colors of the vertices and edges incident to vertices in S remains unchanged. Moreover, the AVD-property always holds for any pair of adjacent vertices u, v such that $u \in S$ and $v \in R$ if $d(u) \neq d(v)$. If $d(u) = d(v)$, where $u \in S$ and $v \in R$, then $C(u) \neq C(v)$ as $C \cap C(u) \neq C \cap C(v)$. Thus, the obtained coloring f is an AVD-total coloring of G which uses $n + k$ colors. Hence $\chi''_a(G) = \Delta + 1$.

Case 2: S has more than one vertex with degree Δ .

In this case, $\chi''_a(G) \geq \Delta + 2$. We will show that k new colors are enough to extend the coloring f to obtain an AVD-total coloring of G . Consider the graph $G' = G - E(G[S])$. Note that the graph G' is a bipartite graph with maximum degree k as $\max\{d(v) : v \in R\} \leq \Delta - n$. Now we know that G' is k -edge colorable, by Theorem 1.3. Let $C' = \{n + 2, \dots, n + k + 1\}$ be a set of k new colors and f' be the k -edge coloring of G' which uses colors from the set C' . Let $f(u_i v_j) = f'(u_i v_j)$, for any edge $u_i v_j \in E(G)$ such that $u_i \in S$ and $v_j \in R$. Finally, color the vertices in R with any available color. Note that the AVD-property is satisfied by any pair of vertices in S . Moreover, the AVD-property always holds for any pair of adjacent vertices u, v such that $u \in S$ and $v \in R$ if $d(u) \neq d(v)$. If $d(u) = d(v)$ where $u \in S$ and $v \in R$, then also $C(u) \neq C(v)$ as $C \cap C(u) \neq C \cap C(v)$. Therefore, the obtained coloring f is an AVD-total coloring of G which uses $n + k + 1$ colors. Hence $\chi''_a(G) = \Delta + 2$. \square

Zhang et al. [20] computed the AVD-total coloring of complete graphs. Consider a complete graph K_n such that $n = 2t + 1$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$. We give an AVD-total coloring g of K_n with $n + 2$ colors such that g has a special property that out of two available colors on each vertex the smaller color is different for at least $n - 1$ vertices. Define $g : V(K_n) \cup E(K_n) \rightarrow \{1, 2, \dots, n + 2\}$ such that

Case 1: $t = 1$, that is, $n = 3$.

$$\text{For } 1 \leq i \leq 3, g(v_i) = i;$$

$$g(v_1 v_2) = 3, g(v_2 v_3) = 4, g(v_3 v_1) = 5.$$

Observe that g is an AVD-total coloring of K_3 using 5 colors. With respect to the coloring g , the set of available colors are $\bar{C}(v_1) = \{2, 4\}$, $\bar{C}(v_2) = \{1, 5\}$ and $\bar{C}(v_3) = \{1, 2\}$.

Case 2: $t = 2$, that is, $n = 5$.

$$\begin{aligned} g(v_1v_2) &= g(v_3v_5) = 1; \\ g(v_1) &= g(v_2v_3) = 2; \\ g(v_5) &= g(v_3v_4) = 3; \\ g(v_2) &= g(v_4v_5) = 4; \\ g(v_3) &= g(v_1v_4) = g(v_2v_5) = 5; \\ g(v_1v_5) &= g(v_2v_4) = 6; \\ g(v_4) &= g(v_1v_3) = 7. \end{aligned}$$

Observe that g is an AVD-total coloring of K_5 using 7 colors. With respect to the coloring g , the set of available colors are $\bar{C}(v_1) = \{3, 4\}$, $\bar{C}(v_2) = \{3, 7\}$, $\bar{C}(v_3) = \{4, 6\}$, $\bar{C}(v_4) = \{1, 2\}$, and $\bar{C}(v_5) = \{2, 7\}$.

Case 3: $t \geq 3$.

$$\begin{aligned} g(v_1) &= n + 2 \text{ and } g(v_i) = i, \text{ for } 2 \leq i \leq n; \\ g(v_{i+j}v_{i+n-j}) &= i, \text{ for } 1 \leq i \leq n, 2 \leq j \leq t; \\ \text{For } 1 \leq i \leq n - 2, \text{ if } i \equiv 1 \text{ or } 2 \pmod{4} &\text{ then } g(v_i v_{i+2}) = n + 1; \\ \text{if } i \equiv 3 \text{ or } 0 \pmod{4} &\text{ then } g(v_i v_{i+2}) = n + 2; \\ g(v_{n-1}v_1) &= n; \text{ and } g(v_n v_2) = 1. \end{aligned}$$

Observe that g is an AVD-total coloring of K_n using $n + 2$ colors. With respect to the coloring g , the set of available colors on each vertex v_i is $\bar{C}(v_i)$ such that $\bar{C}(v_1) = \{1, 2\}$, $\bar{C}(v_2) = \{3, n + 2\}$ and $\bar{C}(v_i) = \{i - 1, i + 1\}$ for $3 \leq i \leq n - 2$. If t is even then $\bar{C}(v_{n-1}) = \{n - 2, n + 2\}$ and $\bar{C}(v_n) = \{n - 1, n + 1\}$. If t is odd then $\bar{C}(v_{n-1}) = \{n - 2, n + 1\}$ and $\bar{C}(v_n) = \{n - 1, n + 2\}$.

For this defined AVD-total coloring g , we have the following observation:

Observation 3.2. *The above defined AVD-total coloring g of K_n , where $n = 2t + 1$, holds the following property:*

For every pair of vertices v_i and v_j such that $\bar{C}(v_i) = \{c_i, d_i\}$ and $\bar{C}(v_j) = \{c_j, d_j\}$; $c_i < d_i$, $c_j < d_j$, $c_i \neq c_j$, and $d_i \neq d_j$, where $i \neq 2$, $j \neq 2$ and $1 \leq i, j \leq n$. Furthermore, $c_2 \neq d_i$ and $d_2 \neq c_i$, for any i , $1 \leq i \leq n$.

Proposition 3.2. *Let $G = (S, R)$ be a split graph such that $|S| = n$, n is odd and $\Delta \geq \max\{d(v) : v \in R\} + n$. If there exists exactly one vertex of degree Δ , then $\chi''_a(G) = \Delta + 1$; otherwise, $\chi''_a(G) = \Delta + 2$.*

Proof. Let $\Delta = n - 1 + \Delta_S$, where Δ_S is the maximum degree of any vertex in S in the bipartite graph $B = G - E(G[S])$. Let $\Delta_S = k$, $S = \{u_1, u_2, \dots, u_n\}$. Since n is odd, $\chi''_a(G[S]) = n + 2$. Let $\mathcal{C} = \{1, 2, \dots, n + 2\}$ be a set of $n + 2$ colors. Consider the AVD-total coloring g of $G[S]$ from Observation 3.2, which uses colors from the set \mathcal{C} . Now we have to extend the coloring g to get an AVD-total coloring of the graph G . Next, we have to color the edges between the sets S and R such that the obtained coloring is an AVD-total coloring.

Case 1: There is only one vertex in S with degree Δ .

Suppose we reorder the vertices of clique S along with the coloring g such that $d(u_2) = \Delta$. Therefore, u_2 has k neighbors in R . Take one edge e incident to u_2 from B and color it with some available color. Consider the graph $G' = G - E(G[S]) - \{e\}$. Note that the graph G' is a bipartite graph with maximum degree $k - 1$. We know that G' is a $(k - 1)$ -edge colorable graph, by Theorem 1.3. Let $\mathcal{C}' = \{n + 3, \dots, n + k, n + k + 1\}$ be a set of $k - 1$ new colors and ϕ be a $(k - 1)$ -edge coloring of G' with colors from the set \mathcal{C}' . Let $M = \{u_{i_1}v_{j_1}, u_{i_2}v_{j_2}, \dots, u_{i_r}v_{j_r}\}$ be a color class with color $n + k + 1$, that is, a matching with respect to the coloring ϕ . Now recolor the edge $u_{i_s}v_{j_s} \in M$ with colors $c_{i_s} \in \mathcal{C}(u_{i_s})$ for all s , $1 \leq s \leq r$. Let $g(u_i v_j) = \phi(u_i v_j)$, for any edge $u_i v_j \in E(G)$ such that $u_i \in S$ and $v_j \in R$. Finally, color the vertices of R with any available color on them. Note that the set of available colors on each vertex u_i is still different, by Observation 3.2. Moreover, the AVD-property always holds for any pair of adjacent vertices u, v such that $u \in S$ and $v \in R$ if $d(u) \neq d(v)$. If $d(u) = d(v)$ where $u \in S$ and $v \in R$, then also $\mathcal{C}(u) \neq \mathcal{C}(v)$ as $\mathcal{C} \cap \mathcal{C}(u) \neq \mathcal{C} \cap \mathcal{C}(v)$. Thus the obtained coloring g is an AVD-total coloring of G using $n + k$ colors. Thus $\chi''_a(G) = \Delta + 1$.

Case 2: S has more than one vertex of degree Δ .

Therefore, $\chi''_a(G) \geq \Delta + 2$, that is equal to $n + k + 1$. We will show that $k - 1$ new colors are enough to extend the coloring g to obtain an AVD-total coloring of G . Consider the graph $G' = G - E(G[S])$. Note that G' is a bipartite graph with maximum degree k . Therefore, G' is a k -edge colorable graph, by Theorem 1.3. Let $\mathcal{C}' = \{n + 3, \dots, n + k, n + k + 2\}$ be a set of k new colors and ϕ be an edge coloring of G' using colors from the set \mathcal{C}' . Similar to Case 1, we consider the matching $M = \{u_{i_1}v_{j_1}, u_{i_2}v_{j_2}, \dots, u_{i_r}v_{j_r}\}$ with the largest color $n + k + 2$ and recolor the edges $u_{i_s}v_{j_s}$ with color $d_{i_s} \in \mathcal{C}(u_{i_s})$ if $i_s = 2$ and c_{i_s} otherwise, for all $1 \leq s \leq r$. Let $g(u_i v_j) = \phi(u_i v_j)$, for any edge $u_i v_j \in E(G)$ such that $u_i \in S$ and $v_j \in R$.

Now color the vertices in R with any available color. Observe that in the obtained coloring, the set of available color on $u_i \in S$ is $\bar{C}(u_i) = \{d_i\}$, for $i \neq 2$, $1 \leq i \leq n$ and $\bar{C}(u_2) = \{c_2\}$. By Observation 3.2, $d_i \neq d_j$ and $c_2 \neq d_i$, for any $1 \leq i, j \leq n$. Thus, the AVD-property holds for each pair of vertices in S . Moreover, the AVD-property always holds for any pair of adjacent vertices u, v such that $u \in S$ and $v \in R$ if $d(u) \neq d(v)$. If $d(u) = d(v)$ where $u \in S$ and $v \in R$, then also $C(u) \neq C(v)$ as $C \cap C(u) \neq C \cap C(v)$. Thus we obtained an AVD-total coloring g of G which uses $n + k + 1$ colors. Hence $\chi''_a(G) = \Delta + 2$. \square

The following result immediately follows from Propositions 3.1 and 3.2:

Theorem 3.1. *Let $G = (S, R)$ be a split graph such that $\Delta \geq \max\{d(v) : v \in R\} + n$, where $|S| = n$. If there exists exactly one vertex of degree Δ , then $\chi''_a(G) = \Delta + 1$; otherwise, $\chi''_a(G) = \Delta + 2$.*

To classify the general split graphs, we have to characterize the split graph G with $\Delta < \max\{d(v) : v \in R\} + n$. We made a little progress in this direction. In the next theorem, we generate an AVD-total coloring with $\Delta + 2$ colors of a split graph G such that $\Delta = \max\{d(v) : v \in R\} + n - 1$.

Theorem 3.2. *Let $G = (S, R)$ be a split graph, where $|S| = n$. If $\Delta = \max\{d(v) : v \in R\} + n - 1$, then there exists an AVD-total coloring of G with $\Delta + 2$ colors.*

Proof. We know that the vertices with degree Δ belong to S only. Let $k = \Delta - (n - 1)$, that is, the maximum number of neighbors in R of any vertex in S . Therefore, $\max\{d(v) : v \in R\} = k$ and $\Delta = n + k - 1$. Now we give an AVD-total coloring of G using $(n + k + 1)$ colors. There are two cases to consider:

Case 1: n is even.

Therefore, $\chi''_a(G[S]) = n + 1$. Let $C = \{1, 2, \dots, n + 1\}$ be a set of $n + 1$ colors and f be an AVD-total coloring of $G[S]$ using colors from the set C . Consider the graph $G' = G - E(G[S])$. Observe that the graph G' is a bipartite graph with maximum degree k . Let $C' = \{n + 2, n + 3, \dots, n + k + 1\}$ be a set of new k colors. Since G' is a k -edge colorable graph by Theorem 1.3, we color the edges of G' from the set C' . Now, the only uncolored vertices left are the vertices in set R . Now color each vertex $v \in R$ with any available color. Note that the AVD-property holds on each pair of adjacent vertices u and v such that $u \in S$ and $v \in R$ because no edge incident to vertex v is colored with the color from the set C while $C(u) \supseteq C$. Thus we get an AVD-total coloring of G using $n + k + 1$ colors.

Case 2: n is odd.

Therefore, $\chi''_a(G[S]) = n + 2$. Let $C = \{1, 2, \dots, n + 2\}$ be a set of $n + 2$ colors and g be an AVD-total coloring of $G[S]$ using colors from the set C with the special property given in Observation 3.2. Consider the graph $G' = G - E(G[S])$ which is a bipartite graph with maximum degree k and so by Theorem 1.3, G' is a k -edge colorable graph. Let $C' = \{n + 3, n + 4, \dots, n + k + 2\}$ be a set of new k colors and ϕ be an edge coloring of G' using colors from the set C' . Suppose that M_1, M_2, \dots, M_k are the color classes of k colors with respect to the coloring ϕ . Note that $|M_i| \leq n$ for $1 \leq i \leq k$. Consider the color class M_k . Note that for every vertex $u \in S$ the set of available colors $\bar{C}(u)$ has two colors from the set C . Now we recolor each edge $e \in M_k$ where $e = uv$ such that $u \in S$ and $v \in R$. Recolor the edge e with the least color in $\bar{C}(u)$. Consider the updated sets of available colors for vertices of S . Observe that $\bar{C}(u_2)$ is equal to either $\bar{C}(u_{n-1})$ or $\bar{C}(u_n)$ and for all other pairs u_i, u_j , $\bar{C}(u_i) \neq \bar{C}(u_j)$. To fix this, we take one edge $e' \in M_i$ such that e' is incident to vertex u_2 for some i , $1 \leq i \leq k - 1$ and recolor the edge e' with the color available in $\bar{C}(u_2)$. It follows that the set $\bar{C}(u_2)$ becomes empty. Now, $g(u_i v_j) = \phi(u_i v_j)$ for all edges $u_i v_j \in E(G)$, where $u_i \in S$ and $v_j \in R$. Now the only remaining uncolored vertices are the vertices of set R . We color each vertex in $v \in R$ with any available color on v . Observe that for the obtained coloring, the AVD-property holds for every pair of vertices of S . Note that the AVD-property holds for every pair of vertices u, v such that $u \in S$ and $v \in R$ because at most one color from the set C belongs to $C(v)$ while $C(u) \supset C$. Thus we obtain an AVD-total coloring of G using $n + k + 1$ colors. \square

We know that if a pair of adjacent vertices with maximum degree Δ exists, then $\chi''_a(G) \geq \Delta + 2$. Therefore, the next corollary follows:

Corollary 3.1. *Let $G = (S, R)$ be a split graph with $|S| = n$ such that $\Delta = \max\{d(v) : v \in R\} + n - 1$. If there exist two vertices with degree Δ in S then $\chi''_a(G) = \Delta + 2$; otherwise $\chi''_a(G) \leq \Delta + 2$.*

4. Classification of complete split graphs

In [3], the total chromatic number of complete split graphs has been characterized. Their result is mainly based on the property of total chromatic number that, for any subgraph H of a graph G , $\chi''(H) \leq \chi''(G)$. However, a similar property does not hold for the AVD-total chromatic number of a graph. For example, for a graph G of order 4 obtained by adding a pendant edge to one of the vertices of K_3 , $\chi''_a(G) = 4$ but $\chi''_a(K_3) = 5$. The violation of this property makes it difficult to characterize the complete split graphs according to their AVD-total chromatic number. This section studies the AVD-total chromatic number of complete split graphs.

Theorem 4.1. Let $G = (S, R)$ be a complete split graph, where $|S| = n, n \geq 2$ and $|R| = m$. If $n + m$ is even, then $\chi''_a(G) = \Delta + 2$.

Proof. If $m = 1$, then G is a complete graph of order $n + 1$ and therefore $\chi''_a(G) = \Delta + 2$, when $n + 1$ is even and $\chi''_a(G) = \Delta + 3$, otherwise. Assume that $m \geq 2$ and the maximum degree of G is $\Delta = n + m - 1$. Therefore, $\chi''_a(G) \geq \Delta + 2 = n + m + 1$. Let $n + m$ be even.

We know that $\chi''_a(K_{n+m}) = n + m + 1$. We take an AVD-total coloring ϕ of a complete graph K_{n+m} using $n + m + 1$ colors. Now define an AVD-total coloring of $G, f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, n + m + 1\}$ such that $f(v) = \phi(v)$, for any vertex $v \in V(G)$ and $f(uv) = \phi(uv)$, for any edge $uv \in E(G)$. Observe that the coloring f is an AVD-total coloring of G as $C_f(v) = C_\phi(v)$ for all $v \in S$ and $C_f(v) \neq C_f(u)$ for any vertices $u \in S$ and $v \in R$ as $d(u) \neq d(v)$. We know that $\chi''_a(G) \geq \Delta + 2$. Thus, $\chi''_a(G) = \Delta + 2 = n + m + 1$. \square

Classification if $(n + m)$ is odd

Next, we further classify the complete split graphs when $n + m$ is odd. Let $G = (S, R)$ be a complete split graph, where $|S| = n$ and $|R| = m$ ($m \geq 2$). Since $m \geq 2, \chi''_a(G) \geq \Delta + 2 = n + m + 1$. Our idea is to construct a Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$, which can be used to obtain an AVD-total coloring of G using $n + m + 1$ colors. In the next result, we show that we can obtain an AVD-total coloring of graph G with $n + m + 1$ colors by using a Latin rectangle having a certain property.

Proposition 4.1. Let $G = (S, R)$ be a complete split graph, where $|S| = n, |R| = m$ ($m \geq 2$), $n < m$ and $n + m$ is odd. Then there exists an AVD-total coloring of G using $n + m + 1$ colors if there exists a Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$ with the following property:

(P) \mathcal{L} contains a subarray A such that A is an idempotent commutative Latin square of size $n \times n$ and at least m columns of the subarray $\mathcal{L} - A$ has at least one missing element from the set $\{n + 1, n + 2, \dots, n + m + 1\}$.

Proof. Let $S = \{u_1, u_2, \dots, u_n\}$ and $R = \{v_1, v_2, \dots, v_m\}$. Suppose that $\mathcal{L} = [l_{i,j}]$ is a Latin rectangle of size $n \times (n + m + 1)$ having the property (P). Let A be an ICLS of size $n \times n$ which is contained in \mathcal{L} and a_j be a missing element from $(n + j)$ -th column such that $a_j \in \{n + 1, \dots, n + m + 1\}$, where $1 \leq j \leq m$. Now we give an AVD-total coloring of G using $n + m + 1$ colors. Define an AVD-total coloring $\phi : E(G) \cup V(G) \rightarrow \{1, 2, \dots, n + m + 1\}$ such that

$$\begin{aligned} \phi(u_i) &= l_{i,i} \text{ and } \phi_{u_i u_j} = l_{i,j}, \text{ for } 1 \leq i, j \leq n; \\ \phi(u_i v_j) &= l_{i,n+j}, \text{ for } 1 \leq i \leq n, 1 \leq j \leq m; \\ \phi(v_j) &= a_j, \text{ for } 1 \leq j \leq m. \end{aligned}$$

Observe that the obtained coloring ϕ is a total coloring. Now we have to prove that the coloring ϕ is an AVD-total coloring. Since $n < m, d(u_i) \neq d(v_j)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Therefore, AVD-property holds for any pair of adjacent vertices $u_i \in S$ and $v_j \in R$. Note that for any pair of vertices u_i, u_j in $S, \bar{C}(u_i) \neq \bar{C}(u_j)$ as $\bar{C}(u_i) = \{l_{i,n+m+1}\}$ which is different for each $i, 1 \leq i \leq n$. Thus, the coloring ϕ is an AVD-total coloring using $n + m + 1$ colors. \square

From the previous result, it is clear that to obtain an AVD-total coloring of the complete split graph G using $n + m + 1$ colors; it is enough to construct a Latin rectangle of size $n \times (n + m + 1)$, having property (P). Observe that the converse of Proposition 4.1 is also true. We can construct a Latin rectangle with the property (P) from an AVD-total coloring of G using $n + m + 1$ colors. Since $n + m$ is odd, there exists an ICLS M of order $n + m$. It is easy to obtain a Latin rectangle of size $n \times (n + m)$ from the ICLS M . Given a Latin rectangle \mathcal{L} of size $n \times (n + m)$ which contains an idempotent commutative subarray A of size $n \times n$, our only job is to expand the size of \mathcal{L} by one column. To achieve this, we define two operations as follows:

Definition 4.1. (Push Operation) An element x is said to be “pushed” to a position $l_{i,j}$ if the element on position $l_{i,j}$ is replaced by x and the elements at positions $l_{i,j}, l_{i,j+1}, \dots, l_{i,n+m}$ are shifted one place to the right.

Definition 4.2. (Swap Operation) An element y on some anti-diagonal D is said to be “swapped” with an element x of an ICLS if the following conditions hold:

1. The element y at two places $l_{i,j}, l_{j,i}$ of anti-diagonal D gets replaced by the element x .
2. The element x gets pushed at two different positions $l_{i,n+k}$ and $l_{j,n+k'}$, where $1 \leq k < k' \leq m + 1$; k and k' are the minimum indexed column in i -th row and in j -th row, respectively such that the following properties hold:
 - (a) the element x is not present in k -th and k' -th columns, and
 - (b) after pushing the element x to the positions $l_{i,n+k}$ and $l_{j,n+k'}$, there does not exist any column C in the resultant subarray $\mathcal{L} - A$ such that no element occurs more than once in C .

In an idempotent commutative Latin square (ICLS), we denote the principal anti-diagonal as D_0 , an upper anti-diagonal as D_i , and the corresponding lower anti-diagonal as D'_i . Observe that an ICLS has the same element on every position of any anti-diagonal; see Fig. 1.

Next, we give an algorithm to construct a Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$ having the property (P). For preparation, we need an ICLS $\mathcal{M} = [m_{i,j}]$ of order $n + m$ such that $m_{i,j} = (i + j) \frac{n+m+1}{2} \pmod{n+m}$. Let \mathcal{M}' be the rectangle obtained from \mathcal{M} by keeping the top n rows. So, \mathcal{M}' is a Latin rectangle which contains an idempotent commutative subarray A of size $n \times n$. Now in order to achieve our goal, we shall append a column to \mathcal{M}' and use the two operations mentioned above to construct \mathcal{L} , which is a Latin rectangle of size $n \times (n + m + 1)$ having the property (P). Let C be a rectangle of size $n \times 1$ with each entry ϕ , where ϕ is a positive integer. Define the Latin rectangle \mathcal{L} as an augmented rectangle obtained by appending C to \mathcal{M}' , i.e., $\mathcal{L} := [\mathcal{M}'|C]$. Note that \mathcal{L} is a Latin rectangle of size $n \times (n + m + 1)$, where the last column contains the element ϕ in each row. We intend to replace the element ϕ in each row with some other element and transform \mathcal{L} such that \mathcal{L} satisfies the property (P). We give an algorithm to achieve our goal as follows:

Algorithm 1

Input: $n, m \in \mathbb{N}$ such that $(n + m)$ is odd, $m \geq 2$ and the Latin rectangle \mathcal{L} .

Objective: Construct a Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$ having the property (P).

Step 1. Define the new element to be added as $n + m + 1$. Fix the index $i = 1$ and $r = n$.

Step 2. We swap the element x present on the anti-diagonal D_i from the positions $m_{r-i,i}$ and $m_{i,r-i}$ and the element y present on the anti-diagonal D'_i from the positions $m_{r,i+1}$ and $m_{i+1,r}$ with the element $n + m + 1$. Mark i -th, $(r - i)$ -th, r -th and $(i + 1)$ -th rows and mark the i -th, $(r - i)$ -th, $(i + 1)$ -th and r -th columns from the rectangle \mathcal{M}' and consider the rectangle \mathcal{M}' with unmarked rows and columns. If the element $n + m + 1$ is placed in every row, then we stop. Otherwise, we repeat this step at most $\frac{m+1-i}{2}$ times, or until the size of the number of unmarked rows become smaller than $(i + 2)$.

Step 3. If the element $n + m + 1$ is placed in every row, then we stop. Otherwise, we repeat Step 2 for different value of i , $2 \leq i \leq m - 1$.

Step 4. If the element $n + m + 1$ is placed in every row, then we stop. Otherwise, we swap the element x present on the principal anti-diagonal D_0 from the positions $m_{r+1-i,i}$ and $m_{i,r+1-i}$ with the element $n + m + 1$, where $i = 1$. Mark i -th and $(r + 1 - i)$ -th rows and mark the i -th and $(r + 1 - i)$ -th columns from the rectangle \mathcal{M}' and consider the rectangle \mathcal{M}' with unmarked rows and columns. We repeat this step for different value of i , $2 \leq i \leq \frac{m+1}{2}$, unless we place the element $n + m + 1$ in at least m or $r - 1$ (whichever is smaller) rows of the rectangle \mathcal{M}' .

Step 5. If the element $n + m + 1$ is placed in every row, then we stop. Otherwise, we push the element $(n + m + 1)$ in one unmarked row \hat{i} such that any element occur at most once in each column of the subarray $\mathcal{M}' - A$ and the element $n + m + 1$ get placed at the position $m_{\hat{i},n+m+1}$.

A natural question here is, whether there exists such a Latin rectangle of size $n \times (n + m + 1)$ or not, when $n + m$ is odd and $m \geq 2$. To check that, we have to count the maximum number of the performed swap or push operations. Next, we give an upper bound on the value of n , for which the above method gives us a Latin rectangle with the property (P). In the following result, we prove that using Algorithm 1, we always obtain a Latin rectangle with property (P) if such a Latin rectangle exists.

Lemma 4.1. *If $n + m$ is odd, $m \geq 2$ and $n \leq m^2 + m + 1$, then Algorithm 1 constructs a Latin rectangle of size $n \times (n + m + 1)$ which contains an idempotent commutative subarray of size $n \times n$. Furthermore, if $n > m^2 + m + 1$ then there does not exist such a Latin rectangle.*

Proof. Observe that the goal of Algorithm 1 is to place the element $n + m + 1$ exactly once into each of the n rows such that each element occurs at most once in each column. Clearly, each swap operation place the element $n + m + 1$ in two different rows, and each push operation place the element $n + m + 1$ in a new row. Let W and P be the maximum number of possible swap and push operations, respectively. Therefore,

$$n \leq 2W + P.$$

We perform a push operation, only to put an element in the columns of $\mathcal{M}' - A$, which has $m + 1$ columns, including the new column. To obtain a Latin rectangle with property (P), at least m columns of the resultant subarray $\mathcal{L} - A$ should have one missing element, which is greater than n . Observe that by performing the maximum number of swap operations, we put every element from the set $\{n + 1, \dots, n + m\}$ into each column of the subarray $\mathcal{M}' - A$. Note that the element $n + m + 1$ can not be placed into the subarray $\mathcal{M}' - A$ by a swap operation. Therefore, to obtain a Latin rectangle with property (P), we can push the element $n + m + 1$ in at most one column of the subarray $\mathcal{M}' - A$. Hence, $P = 1$.

We perform a swap operation to an element on anti diagonals, which would place the new element in two different rows. From each anti-diagonal D_i or D'_i we can perform $\lfloor \frac{m+1-i}{2} \rfloor$ swap operations. Therefore,

$$W = \left\lfloor \frac{m+1}{2} \right\rfloor + 2 \left\lfloor \frac{m}{2} \right\rfloor + 2 \left\lfloor \frac{m-1}{2} \right\rfloor + 2 \left\lfloor \frac{m-2}{2} \right\rfloor + \dots + 2 \left\lfloor \frac{m+1-(m-1)}{2} \right\rfloor$$

$$\mathcal{L} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 6 & 2 & 7 & 3 & 8 & 4 & 9 & 5 & \phi \\ \hline 6 & 2 & 7 & 3 & 8 & 4 & 9 & 5 & 1 & \phi \\ \hline 2 & 7 & 3 & 8 & 4 & 9 & 5 & 1 & 6 & \phi \\ \hline 7 & 3 & 8 & 4 & 9 & 5 & 1 & 6 & 2 & \phi \\ \hline 3 & 8 & 4 & 9 & 5 & 1 & 6 & 2 & 7 & \phi \\ \hline 8 & 4 & 9 & 5 & 1 & 6 & 2 & 7 & 3 & \phi \\ \hline 4 & 9 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & \phi \\ \hline \end{array}$$

Fig. 2. An input Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$ for Algorithm 1, where $n = 7$ and $m = 2$.

Case 1: m is odd.

$$\begin{aligned} W &= \frac{m+1}{2} + (m-1) + (m-1) + (m-3) + \dots + 2(1) + 2(1) \\ &= \frac{m+1}{2} + \frac{(m-1)(m+1)}{2} \\ &= \frac{m(m+1)}{2}. \end{aligned}$$

Case 2: m is even.

$$\begin{aligned} W &= \frac{m}{2} + m + (m-2) + (m-2) + \dots + 2(1) + 2(1) \\ &= \frac{m(m+2)}{2} - \frac{m}{2} \\ &= \frac{m(m+1)}{2}. \end{aligned}$$

Thus, $W = \frac{m(m+1)}{2}$. Therefore, $n \leq m(m+1) + 1 = m^2 + m + 1$. Hence, we can construct a Latin rectangle of size $n \times (n + m + 1)$ having the property (P), if $n \leq m^2 + m + 1$.

On the other hand, suppose that $n > m^2 + m + 1$. Observe that in a Latin rectangle \mathcal{L} of size $n \times (n + m)$ such that it contains an ICLS A of size $n \times n$ as a subarray, and the element present on the principal anti-diagonal D_0 is the only element which is not present in the subarray $\mathcal{L} - A$. So this element can be replaced with the element $n + m + 1$ maximum $\frac{m+1}{2}$ times. Similarly, after D_0 , the elements present on anti-diagonals D_1 and D'_1 are the elements that can be replaced with the element $n + m + 1$ maximum times and so on. That is what we are doing in the above algorithm by using swap and push operations. We have already proved that by using the swap and push operations, we can place the element $n + m + 1$ in at most $m^2 + m + 1$ rows to obtain a Latin rectangle having property (P). Therefore at least one row would be left where the element $n + m + 1$ cannot be placed. Hence it is not possible to place the element $n + m + 1$ in all the n rows in this case. Therefore there does not exist the required Latin rectangle of size $n \times (n + m + 1)$, if $n > m^2 + m + 1$. \square

To demonstrate the execution of Algorithm 1, we give a running example.

For example, we take the Latin rectangle \mathcal{L} given in Fig. 2 with values $n = 7$ and $m = 2$. Now, we will place the new element $n + m + 1 = 10$ in each row by using Algorithm 1. Observe that $n = m^2 + m + 1$.

By Step 2, we perform our first two swap operations on anti-diagonals D_1 and D'_1 and swap the elements marked with circle with the element 10, as shown in Fig. 3.

Note that we cannot perform Step 2 again on any other anti-diagonal. Therefore, we move to Step 4 and perform a swap operation on the principal anti-diagonal in unmarked rows, as shown in Fig. 4.

Note that there is only one unmarked row left. Therefore, we move to Step 5 and place the element “10” in the unmarked row, as shown in Fig. 5. Thus, we get the desired Latin rectangle \mathcal{L} after the execution of Algorithm 1.

Theorem 4.2. Let $G = (S, R)$ be a complete split graph where $|S| = n$ and $|R| = m$. If $n + m$ is odd, $m \geq 2$ and $n \leq m^2 + m + 1$, then $\chi''_q(G) = \Delta + 2$.

Proof. Let $S = \{u_1, u_2, \dots, u_n\}$ and $R = \{v_1, v_2, \dots, v_m\}$. Observe that if $n \leq m$, then $\Delta \geq \max\{d(u) : u \in R\} + n - 1$. Therefore from Theorem 3.1 and Theorem 3.2, $\chi''_q(G) = \Delta + 2$ as $m \geq 2$. We assume that $n > m$. Since $n \leq m^2 + m + 1$, we can construct

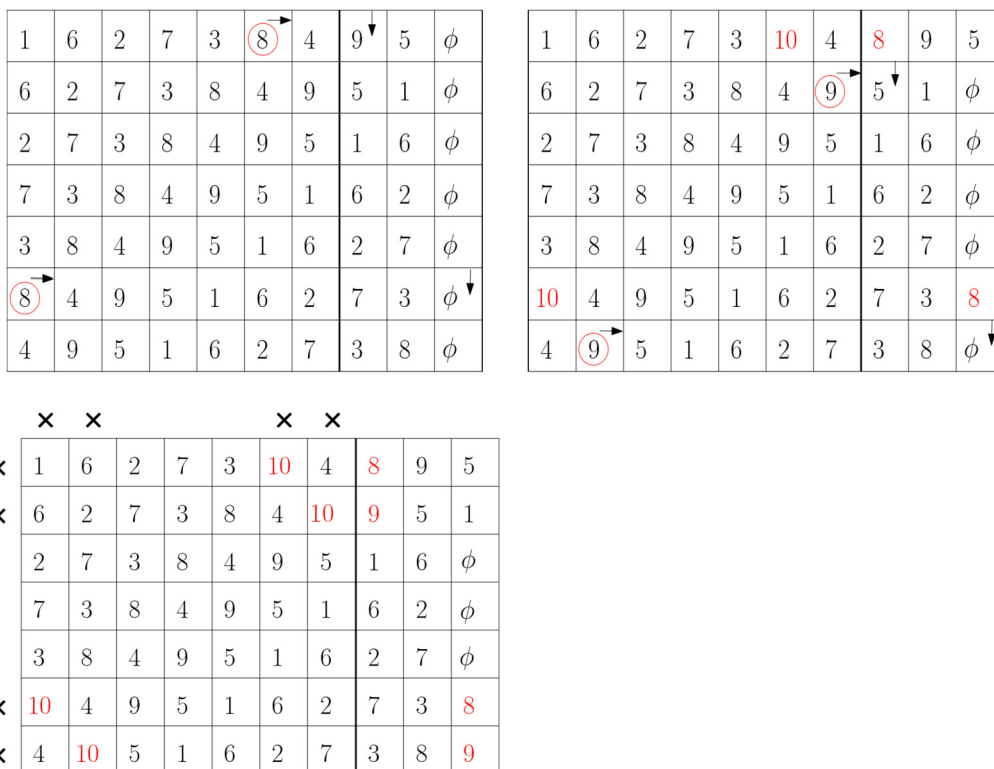


Fig. 3. Swap the elements marked with circle on diagonals with new element 10 by Step 2.

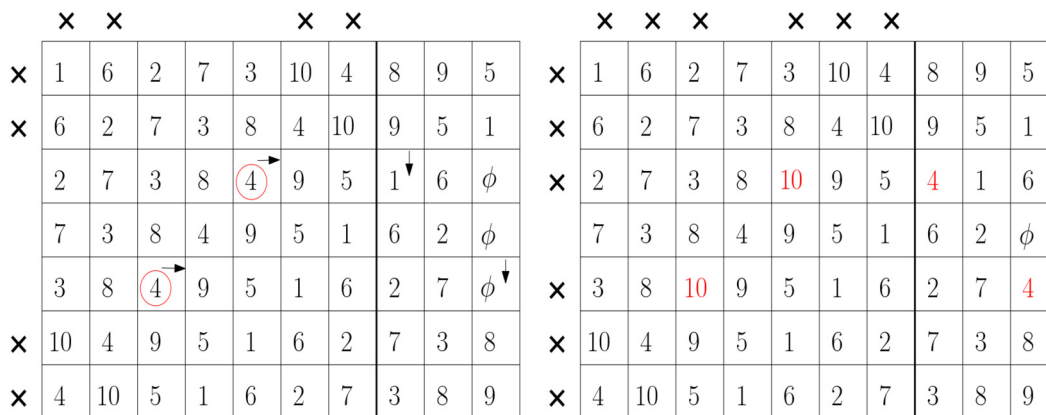


Fig. 4. "Swap" the elements marked with circle on the principal diagonal with new element 10 by Step 4.

a Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$ having the property (P), from Algorithm 1. Therefore, we can construct an AVD-total coloring of G using $n + m + 1$ colors, from Proposition 4.1. Hence, $\chi''_d(G) = n + m + 1 = \Delta + 2$. \square

We know that if $n > m^2 + m + 1$, then we cannot construct a Latin rectangle \mathcal{L} of size $n \times (n + m + 1)$ having the property (P). It implies that if $n > m^2 + m + 1$, $n + m$ is odd and $m \geq 2$, then we cannot give an AVD-total coloring of the complete split graph using $n + m + 1$ colors. Therefore, from the previous results the next corollary characterizes the complete split graphs as follows:

Corollary 4.1. Let $G = (S, R)$ be a complete split graph where $|S| = n$ and $|R| = m$. If $n + m$ is odd, $m = 1$ or $m \geq 2$ and $n > m^2 + m + 1$, then $\chi''_d(G) = \Delta + 3$; otherwise, $\chi''_d(G) = \Delta + 2$.

	×	×	×	×	×	×		×	×	×	×	×	×								
×	1	6	2	7	3	10	4	8	9	5	×	1	6	2	7	3	10	4	8	9	5
×	6	2	7	3	8	4	10	9	5	1	×	6	2	7	3	8	4	10	9	5	1
×	2	7	3	8	10	9	5	4	1	6	×	2	7	3	8	10	9	5	4	1	6
	7	3	8	4	9	5	1	6	2	ϕ		7	3	8	4	9	5	1	6	2	10
×	3	8	10	9	5	1	6	2	7	4	×	3	8	10	9	5	1	6	2	7	4
×	10	4	9	5	1	6	2	7	3	8	×	10	4	9	5	1	6	2	7	3	8
×	4	10	5	1	6	2	7	3	8	9	×	4	10	5	1	6	2	7	3	8	9

Fig. 5. Place the element 10 in unmarked row by Step 5.

5. Conclusion

In this paper, we verify the AVD-total coloring conjecture for split graphs. We also give a characterization for the AVD-total chromatic number of split graphs $G = (S, R)$ such that the maximum degree of the bipartite graph $G - E(G[S])$ belongs to S . Furthermore, we characterize the complete split graphs according to their AVD-total chromatic number. The classification problem for the general split graphs remains unsettled.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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